Department of Mathematics
The University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

November 16, 2013

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
Books, notes, and calculators may be used.
This is a three hour test.

1. Recall that the geometric probability distribution is given by $p(x) = P(X=x) = p(1-p)^{x-1}$ for $x = 1,2,3,\ldots$ for parameter space $p \in (0,1]$.

   a. Why does the parameter space exclude 0 but include 1? What happens when $p = 1$?
   b. Derive the moment generating function for this geometric distribution. Don't forget to include its domain!
   c. Use your answer to part b to derive the mean μ and variance σ^2 for this distribution.

Now imagine that we have an infinite sequence of bowls where bowl x contains x tags numbered $1,2,\ldots,x$. For example, bowl 3 contains 3 tags labeled 1,2 & 3. Define a two-stage selection of a tag as follows: 1) select bowl x with probability p(x) given by the geometric distribution, then 2) select a tag at random (all tags equally likely) from bowl x. Let Y denote the observed value on the tag.

To do parts "d" & "e", you may need to be reminded that the sum of the first n integers is $n(n+1)/2$ and the sum of the squares of the first n integers is $n(n+1)(2n+1)/6$.

   d. Find $E(Y \mid X=x)$ for each $x \geq 1$. What is the random variable $E(Y \mid X)$?
   e. Find $Var(Y \mid X=x)$ for each $x \geq 1$. What is the random variable $Var(Y \mid X)$?
   f. Find E(Y) from your answers above.
   g. Find Var(Y) from your answers above.

2. Say that I have a stack of cards which are either black or white. Call the unknown proportion black "p". I select cards with replacement (WR) until I have selected a total of "m" black cards and let Y denote the number of cards I have to select. In what follows, if the answer is a function, always include the domain!

   a. What is the probability distribution for Y?
   b. Write the likelihood function.
   c. What is a sufficient statistic for the parameter p?
   d. Find the method of moments estimator for p.
   e. Find the maximum likelihood estimator for p.
   f. For testing the null hypothesis $H_0: p = p_0$ versus the alternative $H_A: p \neq p_0$, find the formula for the likelihood ratio test statistic λ (what is it a function of?) and the form of the rejection region in terms of λ.
   g. If m is large, why should the typical chi-square approximation we use for the distribution of a function of λ be of use? Use it to calculate the (approximate) chi-square statistic if $p_0 = ½$, $m = 25$, and $Y = 60$. What is the appropriate degrees of freedom?
   h. How does the answer to part "f" change if we consider instead the one (lower) tailed alternative $H_A: p < p_0$?

3. The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} x, & \text{if } 0 \le x < 1, \\ 2 - x, & \text{if } 1 \le x < 2, \\ 0, & \text{if } x < 0 \text{ or } x \ge 2. \end{cases}$$

(a) Find $\mathrm{E}(X)$.

(b) Find $\mathrm{Var}(2X - 1)$.

(c) If $M_X(t)$ is the moment-generating function of X, find $M_X(0)$, $M'_X(0)$ and $M''_X(0)$.

4. Let $X_1, \ldots, X_n$ denote a random sample from a population with density function given by

$$f(x) = \begin{cases} e^{-(x-\theta)}, & \text{if } x > \theta, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\theta > 0$.

(a) Find a sufficient statistic for θ.

(b) Find the method of moments estimator of θ.

(c) Find the maximum likelihood estimator of θ^2.

(d) Let $X_{(1)} = \min(X_1, \ldots, X_n)$. Find $P(|X_{(1)} - \theta| < 1)$ and $\lim_{n\to\infty} P(|X_{(1)} - \theta| < 1)$.