Department of Mathematics
University of Toledo

Master of Science Degree

Comprehensive Examination

Probability and Statistical Theory

December 6, 2014

Instructions:

Do both problems, all parts
Show all of your computations in your blue book

Prove all of your assertions or quote appropriate theorems

Books, notes, and calculators may be used

Each problem is worth 50 points

This is a three hour test

1. (50 points) $X_1, \cdots, X_n \overset{iid}{\sim} f(x|\theta) = \theta x^{\theta-1}$ for $x \in [0,1]$ and $\theta > 0$.

a. Show that $T = -\sum_{i=1}^{n} \ln X_i$ is a sufficient and complete statistic for θ.
b. Show that T has a Gamma density.
c. Find a moment estimator $\hat{\theta}_{\text{MOM}}$.
d. Find MLE $\hat{\theta}_{\text{MLE}}$.
e. Find the UMVUE $\tilde{\theta}$.
f. Calculate the variance of $\tilde{\theta}$.
g. Calculate the Cramér-Rao Lower Bound. Does the UMVUE reach it?
h. Calculate the asymptotic relative efficiency ARE($\hat{\theta}_{\text{MLE}}$, $\hat{\theta}_{\text{MOM}}$) and compare $\hat{\theta}_{\text{MLE}}$ with $\hat{\theta}_{\text{MOM}}$ according to ARE($\hat{\theta}_{\text{MLE}}$, $\hat{\theta}_{\text{MOM}}$).
i. Use the K-R Theorem to find a UMP level α test $\phi_1(\boldsymbol{x})$ for

$$H_0 : \theta \leq \theta_0 \text{ vs } H_1 : \theta > \theta_0.$$

Write the rejection region R of this test using T and the quantile from a Chi-squared distribution.
j. Find the LRT level α test $\phi_2(\boldsymbol{x})$ for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$. Write the rejection region R of this test using T and the quantile from a Chi-squared distribution.

2. (50 points) Let X be discrete, in fact dichotomous, with $P(X=1) = 1/3$ and $P(X=2) = 2/3$. Further say that given $X = x$, $Y \mid X = x \sim B(x,p)$ where $0 \le p \le 1$. So the conditional distribution of Y given X is binomial with the "n" random, either 1 or 2.

Note: You do not need answers to c – g to be able to do parts h – m.

a. Explicitly write down the conditional distribution of Y given $X = 1$ and the conditional distribution of Y given $X = 2$. Be sure to identify the conditional Range(Y) in each case.

b. Find the marginal distribution of Y and its expectation and variance. What is Range(Y)? Check your answer. Hint: For example, $P(Y=0) = (3 - 5p + 2p^2) / 3$. Show this and the other necessary formulas.

c. Find $E(Y \mid X = 1)$ and $Var(Y \mid X = 1)$.

d. Extend the result in part c to find the probability distribution of the random variable $E(Y \mid X)$. Hint: $Range(E(Y \mid X)) = \{E(Y \mid X=x) : x \in Range(X) \}$. What is this range?

e. Find the probability distribution of the random variable $Var(Y \mid X)$.

f. Perform the calculations that show that the formula $E(E(Y \mid X)) = E(Y)$ works for this example.

g. Perform the calculations that show that the formula $E(Var(Y \mid X)) + Var(E(Y \mid X)) = Var(Y)$ works for this example.

Now assume that we have one observation y from this distribution for Y, i.e. that n = 1 and X is not observed.

h. Write down the likelihood function $L(p\, ; y)$. What is its domain? Note that the formula as a function of p differs depending on the (discrete) value of y.

i. Use your function(s) in part h to find the maximum likelihood estimator $\hat{p}$ as a function of y. Hint: the answer is: $\hat{p}\ (y=0) = 0$, $\hat{p}\ (y=1) = 5/8$ and $\hat{p}\ (y=2) = 1$. Show these.

j. We wish to test H_0: $p = ½$ versus H_A: $p \neq ½$ at level of significance α. Derive the likelihood ratio test statistic $\lambda(y)$. Hint: the answer is $\lambda\ (y=0) = 1/3$, $\lambda\ (y=1) = 24/25$ and $\lambda\ (y=2) = 1/4$. Show these.

k. Say that we decide to reject H_0 in favor of H_A if $y = 0$ or $y = 2$. Show that this is indeed a choice for the likelihood ratio test. For this test, what is a choice for the critical value c for rejecting H_0 when $\lambda \le c$? Hint: there are many correct answers.

l. What is the level of significance α for the test in part k? Note that it is pretty high – how do you explain this? That is, why is it so high?

m. Finally, find and sketch the power function for this test. What is the domain of the power function?