M.S. and M.A. Comprehensive Analysis Exam

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To get full credit you must show all your work.
This exam contains 6 real analysis and 6 complex variables questions.

Real Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. (a) Define the Infimum of a bounded set $A \subset \mathbb{R}$.
   (b) Let $A = \left\{(-1)^n \frac{n-1}{n} : n = 1, 2, 3, \ldots\right\}$. Find $\inf A$. Prove your claim.
2. (a) Define a uniformly continuous function on $\mathbb{R}$.
   (b) Use the definition to show that $f(x) = x^2$ is not uniformly continuous on $[0, \infty)$.
3. (a) Let f be a bounded function on $[a, b]$. Define the Riemann integral $\int_a^b f$.
   (b) Use the definition of Riemann integration to compute $\displaystyle\int_{-1}^{1} f$ for $f(x) = \begin{cases} 0 & -1 \leq x \leq 0 \\ x & 0 < x \leq 1 \end{cases}$.
4. Let $\{x_n\}$ be sequence convergent to zero and $\{y_n\}$ be bounded. Show that $\lim\limits_{n \to \infty} x_n y_n = 0$.
5. Let (X, d) be a metric space and $A, B \subset X$. Show that $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$.
6. Let $(X, d_X), (Y, d_Y)$ be two metric spaces and $f : X \to Y$ be a continuous function. Show that if X is compact, then $f(X)$ is compact.

Complex Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. Find the image of the square $\{z \in \mathbb{C} : 1 < |z| < 2, \mathrm{Im}(z) > 0\}$ under the mapping $\log z$ where log is defined using the principle branch.

2. Let C denote the unit circle with counter-clockwise orientation. Compute the integral
$$\oint_C \bar{z} dz.$$

3. Expand $f(z) = \dfrac{1}{z(z-1)}$ in a Laurent series valid for $\{z \in \mathbb{C} : |z| > 1\}$.

4. Evaluate the integral $\displaystyle\int_0^\infty \frac{1}{(1+x^2)^2} dx$.

5. Find an entire function whose imaginary part is $v(x,y) = x + xy + y + 2015$.

6. Find all possible entire functions f with the property that $|f(z)| \leq |z|$ for all $z \in \mathbb{C}$. Prove that you have found all such functions.