Department of Mathematics
University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

November 16, 1996

Instructions:

Choose five problems from the six given.

All five are worth 20 points.

Show all of your computations in your Blue Book.

Prove all of your assertions or quote appropriate theorems.

Books, notes, and calculators *may be* used.

This is a three hour test.

1. Suppose that the independent random variables $X_1, \ldots, X_n$ have the common distribution $N(1,2)$. Define the random variables $Y_1, \ldots, Y_n$ by

$$Y_i = I_{[X_i > 1]} = \begin{cases} 1, & \text{if } X_i > 1, \\ 0, & \text{if } X_i \leq 1. \end{cases}$$

(a) What is the distribution of Y_1?

(b) Find the distribution of $Y = \sum_{i=1}^{n} Y_i$.

(c) If $n = 4$, find $P(Y = 1)$.

(d) If $n = 4$, find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$.

2. Suppose that $X_1, \ldots, X_n$ is a random sample from a Rayleigh population with density function

$$f(x, \theta) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \qquad x > 0,\ \theta > 0,$$

where θ is an unknown parameter.

(a) Find a complete and sufficient statistic for θ.

(b) Let $T(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i^2$, find $\mathrm{E}(T)$ and $\mathrm{Var}(T)$.

(c) Find the method of moments estimator of θ^4.

(d) Find the maximum likelihood estimator of θ^4.

(e) Find the distribution of $\frac{1}{\theta^2} \sum_{i=1}^{n} X_i^2$.

3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. as $N(\mu, \sigma^2)$ with σ^2 known.

(a) Calculate the Fisher information $I_1(\mu)$ contained in X_1.

(b) Find the maximum likelihood estimator of μ^2.

(c) Find the UMVU estimator of μ^2. Does the UMVU estimator achieve the Cramér-Rao lower bound?

4.

Let $X_1, \ldots, X_n$ be i.i.d. with probability density function given by $f(x) = c/x^2$ for $1<x<\theta$, for $\theta>1$.

a. Find c as a function of θ so that f(x) defines a density for all $\theta>1$.

b. Find and sketch the likelihood function $L(\theta; x_1, \ldots, x_n)$ for $\theta>1$.

c. Find the maximum likelihood estimator for θ.

d. Derive the likelihood ratio test for $H_0:\theta=2$ versus $H_A:\theta\neq2$. Show that it can be written in the form:

Reject H_0 if $\max(X_i) > 2$ or if $\max(X_i) \leq c$ and find the value of c corresponding to $\alpha = .10$.

5.

Let $X_1, X_2, \ldots, X_n$ be independent U[0,a], where a>0, let $Y = \prod_{i=1}^{n} X_i$ and let

$(U,V) = (\min(X_i), \max(X_i))$.

a. Find the distribution (pdf) of Y.
b. Find the joint pdf of (U,V).
c. Find the pdf of the sample range, $R = V-U$.

6.

Let X have one of the following three densities on the interval [0,1].

If D=1, then $f(x) = 2(1-x)$ (decreasing triangular).
If D=2, then $f(x) = 1$ (uniform).
If D=3, then $f(x) = 2x$ (increasing triangular).

Regard D as an unknown parameter and consider the situation where we have only one observation on X.

a. Give the maximum likelihood estimate of D as a function of X.

b. Derive the likelihood ratio test of $H_0:D = 0$ or 1 versus $H_A:D = 2$. Find the critical value if $\alpha = .2$.