December 7, 1999

MA COMPREHENSIVE EXAM IN ALGEBRA

Do **two** problems from each of the three parts. Please give complete proofs. If you do three problems in one of the parts, please indicate which two problems you want graded.

Part A: Groups

1. Let a, b and c be elements of a group G. For each of the following statements, give either a proof or a concrete counterexample.
(a) If a has order 5 and $a^3b = ba^3$, then $ab = ba$.
(b) If $abc = 1$, then $cab = 1$.
(c) If $abc = 1$, then $bac = 1$.

2. If G is a group and H is a subgroup of G of index n, show that G contains a *normal* subgroup K whose index divides $n!$.

3. Show that every group of order 1225 is abelian.

Part B: Rings

4. (a) Show that there is no polynomial $p(x) \in \mathbf{Z}[x]$ with $5p(x) = 1$.
(b) Find a ring R and an explicit ring homomorphism $\phi : \mathbf{Z}[x] \to R$ such that ϕ is onto and such that there exists an element $a \in R$ with $5a = 1$.

5. Let R be a U.F.D. (unique factorization domain) and assume that each of $f(x), g(x) \in R[x]$ has content 1. Prove that $f(x)g(x)$ has content 1.

6. (a) Recall that an ideal P in a commutative ring is *prime* if, whenever $x, y \in R$ such that $xy \in P$, either $x \in P$ or $y \in P$. Show that in $\mathbf{Q}[x]$, every prime ideal is a maximal ideal.
(b) Exhibit a prime ideal in $\mathbf{Z}[x]$ which is *not* maximal.

Part C: Linear Algebra

7. Let U and W be subspaces of the finite dimensional vector space V. Prove that $dimU + dimW = dim(U + W) + dim(U \cap W)$.

8. If V is a finite dimensional inner product space and U is a subspace of V, let $U^{\perp}$ be the orthogonal complement of U. Prove that
(a) $V = U \bigoplus U^{\perp}$
(b) $U^{\perp\perp} = U$.

9. Let V be the space of all polynomials in x of degree at most 3 over $\mathbf{R}$, with the standard basis $B = \{1, x, x^2, x^3\}$. Let T be the linear operator on V defined by

$$T(f) = f + \frac{df}{dx}.$$

Find
(a) the matrix of T with respect to the basis B
(b) the characteristic and minimal polynomials of T
(c) the rank and nullity of T
(d) bases for the range and nullspace of T
(e) all eigenvalues of T.