Spring, 2002

M.S. (Applied Mathematics)
Comprehensive Examination in Analysis

Do five (5) questions from **each** *of Parts A and B.*
Indicate on the front of the blue book which problems you wish to have graded.

R denotes the set of real numbers and **C** the set of complex numbers.

Part A. Real Analysis

1. Suppose $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n$=M where $M \neq 0$. Prove that $\lim_{n\to\infty} a_n/b_n = L/M$.

2. (a) Let $\{f_n\}$ be a sequence of continuous functions on the closed interval $[a,b]$. Define what it means for $f_n \to f$ uniformly.

   (b) Show that f is also continuous on $[a,b]$ when $f_n \to f$ uniformly on $[a,b]$.

   (c) Give an example of continuous functions f_n, f on $[a,b]$ such that $f_n \to f$ point-wise and yet
$$\lim_{n\to\infty} \int_a^b f_n(x)dx \neq \int_a^b f(x)dx.$$

3. (a) Let (X,d) be a metric space. Give two different definitions for X to be compact.

   (b) Prove: If X is compact and A is a closed subset of X, then A is compact.

4. (a) Define the Riemann Integral, $\int_a^b f(x)dx$, for a bounded function f on $[a,b]$.

   (b) Prove that the integral exists if f is a monotonic(non-decreasing)function on $[a,b]$.

5. Let $f_n(x) = (1+x^n)^{1/n}, 0 \leq x \leq 2$. Show that $f_n(x) \to f(x)$ uniformly on $[0,2]$ where $f(x) = 1$ for $0 \leq x \leq 1$ and $f(x) = x$ for $1 < x \leq 2$.

6. Let $f(x)$ be a continuous function on the compact metric space (X,d). Prove that there exists an x_0 where $f(x)$ takes on its maximum value.

7. Let $f(x)$ be a continuous real-valued function on $[a,b]$. Suppose that $f(x) \geq 0$ and that there is one point $c \in [a,b]$ with $f(c) > 0$. Show that $\int_a^b f(x)dx > 0$.

8. Abel's Theorem: Suppose $\sum_{n=0}^{\infty} a_n$ converges. Let $f_n(x) = \sum_{k=0}^{n} a_k x^k$. Show that $f_n(x) \to f(x)$ uniformly on $[0,1]$ where $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Part B. Complex Analysis

1. Write each of the following complex numbers in the form $a + bi$, $a, b \in \mathbf{R}$:

   (a) $(\sqrt{12} - 2i)^6$ (b) $\sin(\frac{\pi}{2} - 3i)$ (c) $\mathrm{Log}(1 - i)$ (d) i^{-i}

   (Log z is the principal branch of the logarithm $\log z$).

2. In each part below, determine if there is an *entire* function $f(z)$ satisfying the stated conditions. If so, give an example. If not, explain why.

   (a) $|f(0)| = 1$ and $|f(z)| = \frac{1}{2}$ for all $z \in \mathbf{C}$ such that $|z| = 1$.

   (b) $f(0) = 1$, $f(1) = 0$ and $|f(z)| \leq 1$ for all $z \in \mathbf{C}$.

3. (a) State the Cauchy-Riemann equations for a complex-valued function $f(z)$ which is differentiable at the point $z_0 = x_0 + y_0 i$.

   (b) If a function $f(z)$ is analytic in a domain D and if $\mathrm{Im} f(z)$ is a constant for all $z \in D$, show that $f(z)$ is a constant in D. Is this true if D is only an open set (rather than a domain)? Justify your answer.

4. Let $u(x, y) = x^3 - 2x - 3xy^2$.

   (a) Show that $u(x, y)$ is harmonic in $\mathbf{R}^2$.

   (b) Find *all* functions which are harmonic conjugates of $u(x, y)$.

   (c) Find an analytic function $f(z)$ such that $u(x, y) = \mathrm{Re}(f(z))$.

5. Let C be the unit circle with center at the origin and with counterclockwise orientation. Define

$$g(\xi) = \oint_C \frac{2z^2 - z + 2}{(z - \xi)^2} dz$$

   for all points ξ not on C. Compute

   (a) $g(2)$ (b) $g(\frac{1}{2})$

6. Compute all possible Laurent series at $z = 0$ for the function

$$f(z) = \frac{1}{z^2 - 3z + 2},$$

   stating *clearly* the domain of convergence in each case.

7. (a) For each of the functions

$$f(z) = \frac{1}{\sin z}, \qquad g(z) = \sin\left(\frac{1}{z}\right), \qquad h(z) = \frac{\sin z}{z},$$

   determine whether $z = 0$ is a *removable singularity*, a *pole*, or an *essential singularity*.

   (b) If C is the unit circle with center at the origin, oriented counterclockwise, use residues to compute

$$\oint_C (z^2 + 1) \sin\left(\frac{1}{z}\right) dz.$$

8. If $a > 0$, use residues to evaluate the improper integral

$$\int_0^\infty \frac{1}{(x^2 + a^2)^2} dz.$$