Department of Mathematics
University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

April 6, 2002

Instructions:
Do all four problems
Show all of your computations in your blue book
Prove all of your assertions or quote appropriate theorems
Books, notes, and calculators may be used
Each problem is worth 25 points
This is a three hour test

1. Let $Y_1, \ldots, Y_n$ be a random sample from a geometric distribution with frequency function

$$P_\theta(Y = y) = (1-\theta)^{y-1}\theta, \qquad y = 1, 2, \ldots, \qquad 0 < \theta < 1.$$

Let $Z_1 = \min(Y_1, Y_2)$, $Z_2 = Y_1 - Y_2$, $Z_3 = \min(Y_1, \ldots, Y_n)$, $Z_4 = \frac{Y_1}{Y_1+Y_2}$, and $Z_5 = Y_1 + Y_2$.

(a) Find the joint distribution of (Z_1, Z_2). Are Z_1 and Z_2 independent? Explain your reasoning.

(b) Find the distribution of Z_1. Also find $E(Z_1)$ and $\mathrm{Var}(Z_1)$.

(c) Find the distribution of Z_3. Also find $E(Z_3)$ and $\mathrm{Var}(Z_3)$.

(d) Find the distribution of Z_4.

(e) Find the joint distribution of (Y_1, Z_5). Are Y_1 and Z_5 independent? Explain your reasoning.

(f) Find the method of moments estimate of θ based on the second moment.

2. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with density function

$$f(x; \theta) = \theta \exp(-\theta x), \qquad x > 0, \quad \theta > 0.$$

(a) Find the maximum likelihood estimate of θ.

(b) Find the Fisher information about θ contained in $(X_1, \ldots, X_n)$. Also find the Cramér-Rao lower bound for the variance of an unbiased estimate of θ.

(c) Find the UMVU estimator of θ. Does the UMVU estimator achieve the Cramér-Rao lower bound? Explain your reasoning.

(d) Use the measure of **mean square error** to compare the maximum likelihood estimator of θ in part (a) with the UMVU estimator of θ in part (c). Which estimator is better? Explain your reasoning.

3. A random sample $X_1, \ldots, X_n$ is drawn from the distribution with the following pdf:

$$f(x;\theta,\nu) = \frac{\theta\nu^{\theta}}{x^{\theta+1}} I_{[\nu,\infty)}(x) \text{ where } \theta>0 \text{ and } \nu>0.$$

   a. Find sufficient statistics for θ and ν.
   b. Find the maximum likelihood estimates of θ and ν.
   c. Consider the test of $H_0:\theta = 1$ versus the two-tailed alternative $H_1:\theta \neq 1$. Let $T = \ln(W/x_{(1)})$ where W denotes the geometric mean $\left(\prod_{i=1}^{n} x_i\right)^{1/n}$ and $x_{(1)}$ denotes the minimum or first order statistic. Find the likelihood ratio λ as a function of T.
   d. Confirm that the likelihood ratio test derived in part c is equivalent to rejecting H_0 if $T<c_1$ or $T>c_2$.
   e. If the observed data were .8, 1.2, 1.2, 1.4, 1.5, 1.7, 1.7, 1.7, 1.9, 2.5, what would be your conclusion? Use level of significance $\alpha = .01$.

4. In this problem we will consider the Wilcoxon Signed Rank test and its power for two different distributions. We will see that the test is independent of the underlying distribution, but the power is not. Let m_X and m_Y denote the medians of the two distributions, and let μ_X and μ_Y denote the means of X and Y. We will test H_0: $m_X = m_Y$ versus the one tailed alternative, H_1: $m_X > m_Y$.
   a. For n=2, that is, with data $\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}$ and $\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}$, find the critical region for the test based on the Wilcoxon signed rank statistic that gives the smallest $\alpha > 0$. What is the level of significance for this test?
   b. Let $\Phi(x)$ denote the CDF for the standard normal distribution. Say that our data $(X,Y)'$ comes from a bivariate normal distribution with mean $(\mu_X, \mu_Y)'$ and covariance matrix $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$. In terms of Φ, find the power of the test from part a as a function of these five parameters. Show that the power is an increasing function of $\Delta = \mu_X - \mu_Y$.
   c. Now say that our data comes from a distribution defined on the unit square $[0,1]\times[0,1]$ with $f(x) = d(ax+by+c)$. Assume that there appropriate constraints on f(x) so that it is always positive on the unit square. Find the constant d and the means μ_X and μ_Y.
   d. Find the power of the test from part a for this distribution. Show that it is an increasing function of $\Delta = \mu_X - \mu_Y$.