M.A. Algebra Comprehensive Exam

April 22, 2003

Do **two** *problems from each of the three parts. Please give complete proofs. If you try three problems in one ot the three parts, please indicate which two problems you wish to be graded.*

Part I

1. Determine whether each of the following statements is true or false. If true, prove it. If false, describe a counterexample.

(a) If G is a group and N is a normal subgroup of G such that both N and G/N are abelian, then G is abelian.

(b) If H is a finite subgroup of a group G and N is a normal subgroup of G such that the index $|G : N|$ is finite and relatively prime to $|H|$, then $H \leq N$.

2. (a) If G is a group and H is a subgroup of G of index $|G : H| = n < \infty$, prove that there is a homomorphism $G \to S_n$ whose kernel is contained in H.

(b) If G is a non-abelian simple group, prove that G has no subgroup of index 3.

3. (a) If G is a cyclic group, prove that the automorphism group of G is abelian.

(b) If G is a group whose automorphism group is cyclic, prove that G is abelian.

(*Hint: Consider the group of inner automorphisms.*)

Part II

4. (a) If F is a field, show that $F[x]$ is a principal ideal domain (P.I.D.).

(b) If D is an integral domain such that $D[x]$ is a P.I.D., show that D is a field.

5. An ideal P of a commutative ring R (with 1) is called a *prime* ideal if, whenever $a, b \in R$ such that $ab \in P$, then either $a \in P$ or $b \in P$.

(a) Prove that any maximal ideal of R is a prime ideal.

(b) Give an example of a commutative ring R and an ideal P of R such that P is prime but not maximal.

6. (a) Show that each of the following polynomials is irreducible over the field indicated:

(i) $x^3 - x + 1 \in \mathbb{Z}_3[x]$, where $\mathbb{Z}_3$ is the ring of integers modulo 3

(ii) $x^4 + x^3 + x^2 + x + 1 = \frac{x^5-1}{x-1} \in \mathbb{Q}[x]$, where $\mathbb{Q}$ is the field of rational numbers

(b) Describe the construction of a field having 27 elements.

Part III

7. Let V be finite-dimensional vector space over the reals, $\mathbb{R}$, and let T be a linear operator on V. Suppose that T commutes with every diagonalizable linear operator on V. Prove that T is a scalar multiple of the identity operator.

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8. (a) Let V and W be vector spaces and let T be a linear operator from V into W. Suppose that V is finite-dimensional. Prove $\text{rank}(T) + \text{nullity}(T) = \dim(V)$.

(b) Let S be the linear operator defined on the space of 3×3 real matrices given by,

$$S(A) = A - A^t,$$

where A^t denotes the transpose of the matrix A. Determine the rank of S.

9. Suppose $T : V \to W$ is a linear map between finite dimensional real vector spaces.

(a) Give the definition of the matrix A_T of T with respect to bases of V and W.

(b) Prove that T is invertible if and only if A_T is invertible.

(c) If $V = W = \{\text{real polynomials of degree} \leq n\}$ and $T = \frac{d}{dx}$ then find the matrix of T with respect to your favorite basis.