THE UNIVERSITY OF TOLEDO
Comprehensive Examination (Analysis) Spring 2003
M.A./M.S. (Pure/Applied Mathematics)

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Part A: REAL ANALYSIS

To pass the exam, answer at least 2 questions from each of the Part A and Part B and 6 questions in total correctly.

1. (a) Define the "uniform convergence" of a sequence $\{f_n\}_{n=1}^{\infty}$ of real-valued functions to the function f on a set E.

   (b) Test whether the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to zero on $[0, \infty)$, where $f_n(x) = \dfrac{nx}{(1+n^2x^2)}$ for $(0 \leq x < \infty)$.

   (c) Prove that a sequence $\{f_n\}_{n=1}^{\infty}$ of real-valued functions on a set E converges uniformly to some function iff given $\varepsilon > 0$, there exists an $N \in I$ such that

$$|f_m(x) - f_n(x)| < \varepsilon \quad (m,\, n \geq N,\, x \in E).$$

2. (a) Let $\{f_n\}_{n=1}^{\infty} \in \mathcal{R}[a,b]$ and $f_n \to f \in \mathcal{R}[a,b]$ pointwise. Then show by an example that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx \neq \int_a^b f(x)dx,$$

   holds.

   (b) Let $\{f_n\}_{n=1}^{\infty} \in \mathcal{R}[a,b]$, $f \in [a,b]$ and $f_n \to f$ uniformly. Then show that $f \in \mathcal{R}[a,b]$ and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

3. Test whether the following integrals are convergent:

   (a) $\displaystyle\int_{-\infty}^{\infty} \left[\frac{(1+x)}{(1+x^2)}\right] dx$

   (b) $\displaystyle\int_{0}^{\infty} \frac{1}{(x^2+\sqrt{x})}\, dx$

   (c) $\displaystyle\int_{0}^{1} \frac{[\log(\frac{1}{x})]}{\sqrt{x}}\, dx$

4. (a) Let (M, ρ) be a metric space. Prove that a subset A of M is totally bounded iff every sequence of points of A contains a Cauchy subsequence.
   (b) If (M, ρ) is a complete metric space and A is a closed subset of M, then show that (A, ρ) is also complete.
   (c) If A is a closed subset of a compact metric space (M, ρ), then prove that (A, ρ) is also compact.

PART B: COMPLEX ANALYSIS

1. Expand $f(z) = \dfrac{1}{z(z-1)}$ in a Laurent series valid for $1 < |z-2| < 2$.

2. Use the Residue Theorem to evaluate

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^3}\, dx.$$

3. Suppose that f is analytic and 1-1 on the open unit disk $\mathbf{D} = \{z \in \mathbf{C} : \ |z| < 1\}$. If $f(z) = \sum_{n=1}^{\infty} a_n z^n$, prove that Area $f(\mathbf{D}) = \pi \sum_{n=1}^{\infty} n|a_n|^2$.

4. (a) If $f(z) = \dfrac{z}{(z^2+2)}$, find the maximum value of $|f(z)|$ for $|z| \leq 1$.
   (b) Suppose that $g(z)$ is entire and that the harmonic function $v(x, y) = \mathrm{Im}[g(z)]$ satisfies $v(x, y) \leq 1$ for all points (x, y) in the xy-plane. Show that $v(x, y)$ must be constant throughout the plane.