1. A pdf is defined by

$$f_{X,Y}(x,y) = \begin{cases} C(x+2y), & 0 < y < 1 \text{ and } 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant C.
(b) Find the marginal pdf $f_X(x)$.
(c) Find the joint cdf $F_{X,Y}(x,y)$.
(d) Find $\mathrm{E}\left[\frac{9}{(X+1)^2}\right]$.

2. Suppose X_i's $(1 \le i \le n)$ are i.i.d with $N(0,\sigma^2)$

$$f(x|\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) \quad (-\infty < x < \infty, \sigma > 0).$$

(a) Find a complete and sufficient statistic for σ^2.
(b) Find the UMVU estimator of σ.
(c) Find a UMP level α test for $H_0 : \sigma^2 \ge \sigma_0^2$ vs $H_a : \sigma^2 < \sigma_0^2$.

3. Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution with σ^2 known. Let $U_i = \frac{X_i - \mu}{\sigma}$ for $i = 1, \ldots, n$.

(a) Are $U_1^2 + U_2^2$ and $\frac{U_1}{\sqrt{U_1^2+U_2^2}}$ independent? Explain your reasoning.

(b) Let $Y = \sin^{-1} \frac{U_1}{\sqrt{U_1^2+U_2^2}}$. Find the distribution of Y. Also find $E(Y)$ and $\mathrm{Var}(Y)$.

(c) Find the distribution of $Z = \tan Y$.

(d) Find the distribution of $V = \frac{U_1}{U_2}$.

(f) Find the uniformly most powerful size α critical region for testing $H_0 : \mu = \mu_0$ against alternatives $H_1 : \mu < \mu_0$.

(g) Find the uniformly most accurate level $1 - \alpha$ U.C.B. for μ.

4. Suppose that $X_1, \ldots, X_n$ $(n \geq 3)$ are independently and identically distributed according to the uniform distribution $U(0, \theta)$.

(a) Find the UMVU estimator of θ^{-1}. Also determine the variance and the mean squared error (MSE) of the UMVU estimator of θ^{-1}.

(b) Find the maximum likelihood estimator (MLE) of θ^{-1}. Also determine the bias, the variance, and the mean squared error (MSE) of the MLE estimator of θ^{-1}.

(c) Compare the performance of the UMVU estimator in part (a) and the maximum likelihood estimator in part (b) in terms of mean squared error.