M.S. Applied Mathematics

April 2004 Comprehensive Exam in Analysis H.L. Bentley, Denis White

3 hours

Instructions: Do 7 questions including at least 2 from Part B on Complex Analysis

Part A: Real Analysis

1. If $\{s_n\}$ is a complex sequence, define its arithmetic means by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \qquad (n = 0, 1, 2, \ldots).$$

   If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

2. Suppose that f is a real function defined on $\mathbf{R}^1$ which satisfies

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

   for every $x \in \mathbf{R}^1$. Does this imply that f is continuous?

3. Suppose that $f : X \to Y$ is a mapping between metric spaces (X, d) and (Y, δ).
   (a) State the definition of *uniform continuity* of f in this setting.
   (b) Suppose that f is continuous and that (X, d) is compact. Show that f is uniformly continuous.

4. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if p=q} \end{cases}$$

   Prove that this is a metric. What subsets of the resulting metric space are open? Which are closed? Which are compact?

5. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

   converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

6. Define

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

   Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$.

7. Let $f : [0,1] \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is not rational} \end{cases}$$

Determine whether or not f is Riemann integrable on [0,1]. If it is then evaluate $\int_0^1 f(x)\,dx$.

Part B: Complex Analysis

Instructions: Do at least 2 questions from Part B

1. Compute all possible Laurent series at $z = 0$ for the function.

$$f(z) = \frac{1}{z^2 - z - 2}$$

Specify the domain of convergence of each series.

2. Let $u(x, y) = x^3 + 2xy - 3xy^2$.
   (a) Show that u is harmonic.
   (b) Find all harmonic conjugates of u.
   (c) Find an analytic function $f(z)$ so that $u(x, y) = \Re f(x + iy)$.

3. Use the residue theorem to evaluate $\displaystyle\int_0^{\infty} \frac{1}{(x^2+4)^2}\,dx$