M.A. Algebra Comprehensive Exam

April 16, 2005

General Instructions: Do **two** problems from each of the three parts. Be sure to give complete proofs, and be sure to clearly indicate which problems you wish to have graded.

Part I

1. Let $Z(G) = \{z \in G \mid zg = gz \ \forall g \in G\}$ be the center of the group G. Verify that $Z(G) \trianglelefteq G$. Prove that if $G/Z(G)$ is cyclic then G must be abelian.

2. Use the multiplicative group $(\mathbf{Z}/p\mathbf{Z})^*$ to prove Fermat's Little Theorem:

$$\text{If } p \text{ is prime then } a^p \equiv a \pmod{p} \ \forall a \in \mathbf{Z}.$$

3. Suppose that G is a group.
   (a) Prove that if $H \leq G$ has index two then $H \triangleleft G$.
   (b) Use part (a) to prove that the alternating group A_4 does not have a subgroup with six elements.

Part II

4. (a) Let R be a ring with no zero divisors. Show that R has the left and right cancellation properties: if a, b, $x \in R$ and $x \neq 0$, then

$$ax = bx \to a = b \text{ and } xa = xb \to a = b.$$

   (b) Give the definition of *integral domain*.
   (c) Show that a *finite* integral domain is actually a field.

5. Let R be a commutative ring. An element $x \in R$ is called *nilpotent* if $x^n = 0$ for some $n \in \mathbf{Z}$.

   (a) Prove that the set of nilpotent elements in R forms an ideal, called the *nilradical* and denoted $\eta(R)$.

   (b) Prove that $\eta(R/\eta(R)) = 0$.

6. Write down all monic irreducible polynomials of degree 3 in the ring $Z_3[x]$.

Part III

7. Let

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

   (a) Find the characteristic polynomial of A.

   (b) Find all the eigenvalues and for each find a basis of the corresponding eigenspace.

   (c) Is A diagonalizable? Explain.

8. Suppose V, W, X are vector spaces of dimensions 5, 4, 5 respectively. Let $T : V \to W$ and $S : W \to X$ be linear transformations. Can the composite map $ST : V \to X$ be an isomorphism? If so give an example, if not prove it.

9. Let $P_3(x)$ be polynomials with real coefficients of degree ≤ 3 and $P_5(x)$ be those with degree ≤ 5. Define $T : P_3(x) \to P_5(x)$ by:

$$T(p(x)) = (x^3 + x - 1)p'(x).$$

   (a) Prove that T is a linear transformation.

   (b) Write down bases for $P_3(x)$ and $P_5(x)$.

   (c) Write down the matrix for T in terms of your bases.

   (d) Find bases for the range and nullspace of T and verify that the rank-nullity theorem holds.