

**THE UNIVERSITY OF TOLEDO**  
**Comprehensive Examination (Analysis)**  
**Spring 2005**  
**M.A. Pure Mathematics**

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*Solve any three questions from each of the parts A & B. 100% will be awarded for complete answers to three questions in each part. To obtain full credit you must show all of your work. Indicate clearly which six problems you wish to have graded.*

**PART A: REAL ANALYSIS**

1. i) State the definition of convergence for a sequence of real numbers.  
 ii) Prove that if  $\lim_{n \rightarrow \infty} a_n = a$  and  $k$  is any real number then  $\lim_{n \rightarrow \infty} ka_n = ka$ .  
 iii) If  $a_0 = 0$  and  $a_{n+1} = \frac{1+2a_n}{2+a_n}$  prove that the sequence  $\{a_n\}$  is convergent and find the limit. State carefully any results that you quote in your proof.
  
2. i) State the definition of continuity for a real-valued function whose domain is a subset of  $\mathbb{R}$ .  
 ii) Prove from the definition in (i) that the function  $f(x) = x^2$  is continuous at  $x = 2$ .  
 iii) Suppose that  $g(x)$  is a twice-differentiable odd function, that is,  $g(-x) = -g(x)$  defined on an open interval of  $\mathbb{R}$  that contains 0. Find  $g''(0)$  and justify your answer.  
 [Hint: What do you think can be said about  $g'(x)$ ? Look at some examples.]
  
3. i) Prove that a convergent sequence of real numbers is bounded.  
 ii) Suppose  $\lim_{n \rightarrow \infty} a_n = \ell$  and  $\lim_{n \rightarrow \infty} b_n = \ell$  and for each positive integer  $n$ ,  $a_n \leq c_n \leq b_n$ . Prove  $\lim_{n \rightarrow \infty} c_n = \ell$ .
  
4. i) State Rolle's Theorem.  
 ii) Examine whether the hypotheses and conclusions of Rolle's Theorem hold for  $f(x)$  defined by  $x \sin\left(\frac{1}{x}\right)$  for  $x \neq 0$  and  $f(0) = 0$  on the interval  $\left[\frac{-1}{2\pi}, \frac{1}{2\pi}\right]$ .  
 iii) If  $a_0, a_1, \dots, a_n$  are real numbers such that  $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \dots + \frac{a_n}{n+1} = 0$  prove that the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$$

has a solution between 0 and 1.

5. i) Prove that if  $f$  is integrable on  $[a, b]$  and  $m \leq f(x) \leq M$  for all  $x \in [a, b]$  then there exists  $\mu$  such that  $m \leq \mu \leq M$  and  $\int_a^b f = \mu(b - a)$ .

ii) Prove that if  $f$  is continuous on  $[a, b]$  then  $f$  is integrable on  $[a, b]$ .

[Hint: One way is to use the facts that  $f$  is bounded and uniformly continuous on  $[a, b]$ .]

## PART B: COMPLEX ANALYSIS

1. Derive:

a) The Taylor series expansion of  $f(z) = \frac{1}{1-z}$  about the point  $z = i$  and indicate the radius of convergence.

b) The Laurent series expansion of  $f(z) = \frac{1}{(1-z)(3-z)}$  for the domain  $1 < |z| < 3$ .

2. Let  $f(z) = \begin{cases} \frac{(\bar{z})^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$

a) Show that  $f'(0)$  does not exist.

b) Let  $u$  and  $v$  denote the real and imaginary components of  $f$ . Show that  $u$  and  $v$  satisfy the Cauchy-Riemann equations at  $z = 0$ .

3. Find the image of:

a) the vertical line  $x = a$ ,  $a > 0$  under the map  $f(z) = z^2 + 1$ . Draw the image in the  $uv$ -plane and indicate the orientation.

b) the infinite strip  $\frac{\pi}{4} \leq y \leq \frac{\pi}{2}$  under the map  $f(z) = e^{2z}$ .

4. Evaluate the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)(x^2 + 9)} dx.$$

5. i) Let  $f(z) = u(x, y) + iv(x, y)$  be analytic in some domain  $D$  in  $\mathbb{C}$ . Show that  $u$  and  $v$  satisfy the Cauchy-Riemann equations in  $D$ .

ii) Find an analytic function on  $\mathbb{C}$  whose real part is

a)  $x^3 - 3xy^2$

b)  $\exp(x) \cos y$