MS COMPREHENSIVE EXAM
DIFFERENTIAL EQUATIONS
SPRING 2005

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This exam has two parts, ordinary differential equations and partial differential equations. Choose three problems from each part.

Part I: Ordinary Differential Equations

1. Find the general solution to the system:

$$\begin{cases} \dot{x}_1(t) = x_1(t) \\ \dot{x}_2(t) = x_2(t) \end{cases} .$$

2. Assume you have the following differential equation

$$y'(x) = f(x) + g(x)y(x) + h(x)y^2(x),$$

where f, g, and h are continuous. Show that if $y_1(x)$ is a particular solution, then the transformation

$$y(x) = y_1(x) + \frac{1}{w(x)}$$

reduces the original equation to a new first-order linear equation

$$w'(x) + [g(x) + 2y_1(x)h(x)]w(x) = -h(x).$$

3. Find all the equilibrium points for each system and classify the behavior of solution trajectories near the equilibrium point. Indicate direction of increasing t. Sketch.

(a) $\dfrac{dx}{dt} = y$

$\dfrac{dy}{dt} = x + y^2$

(b) $\dfrac{dx}{dt} = x + y^2$

$\dfrac{dy}{dt} = x^2 - y$

4. In each of the following, find the form of a particular solution to the given non-homogeneous equation. Do not evaluate the constant, just write down the form of the particular solution.

   (a) $y^{(iv)} + y''' - 2y' = t + te^{t}$

   (b) $y^{(vi)} - 2y^{(v)} = t^2 + 1 + 2e^{-\frac{1}{2}t}$

Part II: Partial Differential Equations

1. Let A be a linear operator such that if u is any real-valued function then Au is real-valued. Show that if $Au = 1$ has a complex-valued solution, then it also has a real-valued solution.

2. Solve the following problem for the heat equation:

$$\begin{aligned} u_t &= u_{xx} && 0 \le x \le \pi,\ 0 < t < \infty \\ u(0,t) &= u(\pi,t) = 0 && 0 < t < \infty \\ u(x,0) &= \sin 3x && 0 \le x \le \pi \end{aligned}$$

   Find the behavior of the heat for large values of t.

3. Solve the following initial value problem.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} 1 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} u_1(x,0) &= \sin x \\ u_2(x,0) &= \cos x \end{aligned}$$

4. Solve the given initial value problem and determine the values of x, y and z for which it exists:

$$u_x + u_y + zu_z = u^3, \qquad u(x,y,1) = h(x,y).$$