Master of Arts Comprehensive Analysis Exam Spring 2006

Real and Complex Analysis

To obtain full credit you must show all your work. Indicate clearly which three questions you wish to be graded.

Part 1. Real Analysis. 100% will be obtained four *complete* answers to three questions.

1. (i) State the definition of convergence of a sequence of real numbers.

(ii) Prove that if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty}(a_n + b_n) = a + b$.

(iii) Prove that if $-1 < x < 1$, then $\lim_{n\to\infty} x^n = 0$.

2. (i) A sequence of real numbers x_n is said to be non-decreasing if $p > n$ implies that $x_p \geq x_n$. Prove that a bounded non-decreasing sequence of real numbers is convergent.

(ii) Define a sequence $\{x_n\}$ by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{(2 + \sqrt{x_n})}$ for $n = 1, 2, 3 \cdots$. Prove that $\{x_n\}$ is convergent and find the limit.

3. (i) State the definition of continuity for a real-valued function whose domain is a subset of $\mathbb{R}$.

(ii) Prove that a real-valued function $f(x)$ whose domain is a subset of $\mathbb{R}$ is continuous at $x \in X$ if for all sequences $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x$ we have that $\lim_{n\to\infty} f(x_n) = f(x)$.

(iii) Consider the function

$$f(x) = \begin{cases} x^2, & x \geq 0, \\ -x^2, & x < 0. \end{cases}$$

Discuss the existence and continuity of f and its first two derivatives at all points of $x \in \mathbb{R}$. Justify your answers but you do not need to prove standard theorems.

4. (i) State Rolle's Theorem.

(ii) Suppose that an $\mathbb{R}$-valued function $f(x)$ is differentiable at $x = a$ and $f(x) \leq f(a)$ for all x in some open interval containing a. Prove that $f'(a) = 0$.
[Hint: It is sufficient to consider left hand and right hand limits].

(iii) Deduce Rolle's Theorem from (ii).

5. (i) Prove from the definition that the Riemann integral $\int_0^1 (2x+1)dx$ exists.

(ii) Decide whether the following series converge and justify your answers briefly:
$\sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}$, $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$, $\sum_{n=1}^{\infty} \frac{\ln n^2}{n}$.

Part 2. Complex Analysis. 100% will be obtained for *complete* answers to four questions. Indicate clearly which three questions you wish to be graded.

1. (i) Evaluate completely $(1+i)^{(1-i)}$.

(ii) Determine either from the definition or by means of the Cauchy-Riemann equations the points in $\mathbb{C}$ where the function $f(z) = z\bar{z}$ is differentiable as a complex function.

2. (i) Define $f(z) = e^{\left(\frac{1}{z}\right)}$ $(z \neq 0), f(0) = 0$. At what points is $f(z)$ differentiable as a complex function? Justify your answer.

(ii) The complex numbers z_1, z_2 and z_3 are the vertices of an isosceles right-angled triangle with the right angle at z_3. Prove that $(z_1 - z_3)^2 = 2(z_1 - z_3)(z_2 - z_3)$.

3. (i) Find a complex analytic function whose real part is $e^x(x\cos(y) - y\sin(y))$.
(ii) Show that the Möbius transformation T preserves the cross ratio: that is if $w_i = T(z_i)$ $(1 \leq i \leq 4)$ then $\frac{(z_1-z_4)(z_2-z_3)}{(z_1-z_2)(z_3-z_4)} = \frac{(w_1-w_4)(w_2-w_3)}{(w_1-w_2)(w_3-w_4)}$.

4. Evaluate by any valid method:

(i) $\int_0^{2\pi} \frac{dx}{a+\cos(x)}$.

(ii) By integrating $\frac{e^{iz}}{z}$ over a suitable contour evaluate $\int_0^{\infty} \frac{\sin(x)dx}{x}$.

5. (i) Let $f(z) = \frac{z^3+2z^2+4}{(z-1)^3}$. Find the Laurent expansion for $f(z)$ about the singular point $z = 1$. For what values of z is this expansion valid?
(ii) Let C be a piecewise smooth closed contour in $\mathbb{C}$. Suppose that $f(z)$ is a analytic inside and on C and that a and b are distinct points inside C. Show that $\int_C \frac{f(z)dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b}(f(a) - f(b))$.