Master of Science Comprehensive Analysis Exam Spring 2006

Real and Complex Analysis

To obtain full credit you must show all your work

Part 1. Real Analysis. 100% will be obtained for *complete* answers to four questions. Indicate clearly which four questions you wish to be graded.

1. (i) State the definition of convergence of a sequence of real numbers.

(ii) Prove that if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $\lim_{n\to\infty}(a_n + b_n) = a + b$.

(iii) Prove that if $-1 < x < 1$, then $\lim_{n\to\infty} x^n = 0$.

2. (i) A sequence of real numbers x_n is said to be non-decreasing if $p > n$ implies that $x_p \geq x_n$. Prove that a bounded non-decreasing sequence of real numbers is convergent.

(ii) Define a sequence $\{x_n\}$ by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{(2 + \sqrt{x_n})}$ for $n = 1, 2, 3, \cdots$. Prove that $\{x_n\}$ is convergent and find the limit.

3. (i) State the definition of continuity for a real-valued function whose domain is a subset of $\mathbb{R}$.

(ii) Prove that a real-valued function $f(x)$ whose domain is a subset of $\mathbb{R}$ is continuous at $x \in X$ if for all sequences $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x$ we have that $\lim_{n\to\infty} f(x_n) = f(x)$.

(iii) Consider the function

$$f(x) = \begin{cases} x^2, & x \geq 0, \\ -x^2, & x < 0. \end{cases}$$

Discuss the existence and continuity of f and its first two derivatives at all points of $x \in \mathbb{R}$. Justify your answers but you do not need to prove standard theorems.

4. (i) State Rolle's Theorem.

(ii) Suppose that an $\mathbb{R}$-valued function $f(x)$ is differentiable at $x = a$ and $f(x) \leq f(a)$ for all x in some open interval containing a. Prove that $f'(a) = 0$.
[Hint: It is sufficient to consider left hand and right hand limits].

(iii) Deduce Rolle's Theorem from (ii).

5. (i) Let X be a non-empty subset and define a map $d : X \times X \mapsto \mathbb{R}$ such that for all $x, y \in X$, $d(x, y) = 2$ if $x \neq y$ and for all $x \in X$, $d(x, x) = 0$. Prove in detail that (X, d) is a metric space.

(ii) Prove that the sequence of points $x_n = \frac{n}{n+1}$, $n \in \mathbb{N}$ is a Cauchy sequence.

(iii) Let (X, d) be a metric space and let $Y \subset X$. Show that a subset $A \subset Y$ is open in Y if and only if there exists a subset B open in X such that $A = B \cap Y$.

6. (i) Prove from the definition that the Riemann integral $\int_0^1 (2x + 1) dx$ exists.

(ii) Decide whether the following series converge and justify your answers briefly: $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$, $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$,$\sum_{n=1}^{\infty} \frac{\ln n^2}{n}$.

7. Suppose that (X, d) is a complete metric space and T is a map from X to itself such that for all $x, y \in X$ $d(T(x), T(y)) \leq kd(x, y)$ where $0 < k < 1$. Prove that there is a unique $x_0 \in X$ such that $T(x_0) = x_0$.

Part 2. Complex Analysis. 100% will be obtained for *complete* answers to four questions. Indicate clearly which four questions you wish to be graded.

1. (i) Evaluate completely $(1+i)^{(1-i)}$.

(ii) Determine either from the definition or by means of the Cauchy-Riemann equations the points in $\mathbb{C}$ where the function $f(z) = z\bar{z}$ is differentiable as a complex function.

2. (i) Define $f(z) = e^{(\frac{1}{z})} \quad (z \neq 0), f(0) = 0$. At what points is $f(z)$ differentiable as a complex function? Justify your answer.

(ii) The complex numbers z_1, z_2 and z_3 are the vertices of an isosceles right-angled triangle with the right angle at z_3. Prove that $(z_1 - z_3)^2 = 2(z_1 - z_3)(z_2 - z_3)$.

3. (i) Find a complex analytic function whose real part is $e^x(x\cos(y) - y\sin(y))$.
(ii) Show that the Möbius transformation T preserves the cross ratio: that is if $w_i = T(z_i) \quad (1 \leq i \leq 4)$ then $\frac{(z_1-z_4)(z_2-z_3)}{(z_1-z_2)(z_3-z_4)} = \frac{(w_1-w_4)(w_2-w_3)}{(w_1-w_2)(w_3-w_4)}$.

4. Evaluate by any valid method:

(i) $\int_0^{2\pi} \frac{dx}{a+\cos(x)}$.

(ii) By integrating $\frac{e^{iz}}{z}$ over a suitable contour evaluate $\int_0^{\infty} \frac{\sin(x)dx}{x}$.

5. (i) Let $f(z) = \frac{z^3+2z^2+4}{(z-1)^3}$. Find the Laurent expansion for $f(z)$ about the singular point $z = 1$. For what values of z is this expansion valid?
(ii) Let C be a piecewise smooth closed contour in $\mathbb{C}$. Suppose that $f(z)$ is a analytic inside and on C and that a and b are distinct points inside C. Show that $\int_C \frac{f(z)dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b}(f(a) - f(b))$.

6. (i) Evaluate $\int_C (2z^2 + 3z + 5 + \frac{(z^4+2z+1)}{(z-2)} + \frac{1}{(2z-1)})dz$ where C is the unit circle center at the origin in $\mathbb{C}$.

(ii) Compute the residues of the following functions at the points indicated:
$\frac{1}{z^2+1}$ at $z = i$, $\frac{1}{(z-\sin(z))}$ at $z = 0$.

7. (i) Find a Mobius transformation that maps 1, 2 and i to i,∞ and 1, respectively.

(ii) Evaluate $\int_C \frac{(z^4+2z+1)dz}{(z-2)^4}$ where C in any piecewise smooth closed contour containing the point $z = 2$.