# MS COMPREHENSIVE EXAM
# DIFFERENTIAL EQUATIONS
# SPRING 2006

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*This exam has two parts, ordinary differential equations and partial differential equations. Choose three problems in each part. Mark clearly the problems you choose and show the details of your work.*

## Part A: Ordinary Differential Equations

1. If $u(x)$ and $v(x)$ are solutions of

$$[r(x)y'(x)]' + p(x)y(x) = 0$$

show that

$$r(x)[u(x)v'(x) - v(x)u'(x)] = k$$

where $k$ is constant.

2. Solve the Sturm-Liouville system and show that the first three eigenvalue-eigenfunction pairs satisfy the system.

$$\begin{aligned} y''(x) + \lambda y(x) &= 0 \\ y(0) = y(\pi) &= 0. \end{aligned}$$

3. (a) Locate all equilibrium points for the given autonomous system. Determine whether the equilibrium point or points are asymptotically stable, stable but not asymptotically stable or unstable.

$$\begin{cases} y_1' = 2y_1 + y_2 + y_3 \\ y_2' = y_1 + y_2 + 2y_3 \\ y_3' = y_1 + 2y_2 + y_3 \end{cases}$$

(b) Perform a stability analysis of the following system at the equilibrium point $(0, 0)$ and determine the stability properties of the system at this point.

$$\begin{cases} x' = \frac{1}{2}\,(1 - \frac{1}{2}\,x - \frac{1}{2}\,y)x \\ y' = \frac{1}{4}\,(1 - \frac{1}{3}\,x - \frac{2}{3}\,y)y. \end{cases}$$

4. (a) Find the general solution of

$$y'' - 2y' + y = e^t$$

(b) The following function $y(t)$ is the solution of $y'' + \alpha y' + \beta y = g(t)$, $y(0) = y_0$, $y'(0) = y_0'$. Determine the constants $\alpha$, $\beta$, $y_0$ and $y_0'$.

$$y(t) = e^{-t} + \int_0^t \frac{e^{t-\lambda} - e^{-(t-\lambda)}}{2}\, g(\lambda)d\lambda$$

**Part B: Partial Differential Equations**

1. Find the Fourier series for $f(x)$ defined as

$$f(x) = 1, \quad -\pi < x < 0$$
$$f(x) = 2, \quad 0 < x < \pi.$$

sketch $f(x)$.

2. Find a particular solution of the equation by i) exponential substitution, and ii) separation of variables.

$$u_x + 2u_y - u = 0$$

3. Solve

$$x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = xt;$$

$u(x, 0) = 0$, $u(0, t) = 0$, $x, t \geq 0$ by using the Laplace transform.

4. Solve $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$, $0 < x < 3$, $t > 0$ given that $u(0, t) = u(3, t) = 0$, $u(x, 0) = 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x$. $u$ is bounded for $0 < x < 3$, $t > 0$.