MA exam: algebra, 21 April 2007

Please do four problems, including one from each of the three sections. Give complete proofs — do not just quote a theorem. Please indicate clearly which four problems you want to be graded.

Part I: Group theory

1. Let G be a cyclic group of order n and let d be a positive divisor of n.

a. Show that G has an element of order d.

b. Suppose that x and y are elements of G of order d. Prove that there is an integer m which is relatively prime to n such that $y = x^m$.

2. Let $\phi\colon G \to H$ be a group homomorphism which is onto.

a. Suppose that A is a normal subgroup of G. Prove that $\phi(A)$ is a normal subgroup of H.

b. Suppose that A is a subgroup of G with the property that $\phi(A)$ is a normal subgroup of H. Is it true that A must be a normal subgroup of G? Either prove this is true or else provide a counterexample.

Part II: Ring theory

3. Let p be a prime integer and set

$$R = \left\{ \frac{m}{n} \;\middle|\; m \text{ and } n \text{ are integers and } p \text{ does not divide } n \right\}.$$

You may assume that R is a ring under the usual operations on fractions.

a. Find all of the units of R.

b. Show that each nonzero element of R has the form up^k where u is a unit of R and $k \geq 0$.

c. Show that each ideal of R is a principal ideal.

d. Find all of the irreducible elements in R.

4. *a.* Prove that a euclidean domain is a principal ideal domain.

b. Is $\mathbb{R}[x, y]$ a euclidean domain? (Here, $\mathbb{R}$ denotes the field of real numbers.) Either prove that it is or else provide a detailed explanation of why it is not.

Part III: Linear algebra

5. Let V be the set of all $n \times n$ matrices over a field F. Let I denote the identity matrix. You may assume that V is a vector space over F, under the usual matrix operations.

a. Give a basis for V.

b. Let $A \in V$. Prove that there is a positive integer m and scalars $\alpha_0, \ldots, \alpha_{m-1} \in F$ such that

$$A^m = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_{m-1} A^{m-1}.$$

c. Let $A \in V$. Show that A is invertible if and only if A satisfies an equation as above with $\alpha_0 \neq 0$.

6. Let A be the 3×3 real matrix

$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

a. Find the characteristic equation of A and the eigenvalues of A.

b. Find a basis for each eigenspace of A.

c. Is A diagonalizable? If it is, diagonalize it. If not, explain why not.