MA exam: algebra, 19 April 2008

Please do four problems, including one from each of the three sections. Give complete proofs — do not just quote a theorem. Please indicate clearly which four problems you want to be graded.

Part I: Group theory

1. Let G be the symmetric group of degree 11, and let $g_1 = (1, 3, 9)(2, 8, 5, 7)$, $g_2 = (2, 5)(3, 4, 7, 11, 6, 8)$.

a. Compute $|g_1|$ and $|g_2|$.

b. Compute $g_1 g_2 g_1^{-1}$.

c. Are g_1 and g_2 conjugate in G? Prove your assertion.

2. Suppose that G is a finite group, and $\phi\colon G \to H$ is a group homomorphism. Let p be a prime.

a. Prove that if S is a p-subgroup of G then $\phi(S)$ is a p-subgroup of H.

b. Prove that if ϕ is surjective (onto) and S is a Sylow p-subgroup of G then $\phi(S)$ is a Sylow p-subgroup of H.

Part II: Ring theory

3. Let $R = \{a + 3bi \mid a, b \in \mathbb{Z}\}$.

a. Prove that R is an integral domain.

b. Prove that R is not a unique factorization domain.

4. Let R be the subset of 2×2 real matrices which commute with the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

a. Prove that R is a ring.

b. Prove that $R \cong \mathbb{R}[x]/I$, where I is the ideal generated by x^2.

Part III: Linear algebra

5. Let V and W be be finite-dimensional vector spaces over the field K.

a. Prove that if $T\colon V \to W$ is a linear transformation then $\ker(T)$ is a subspace of V. [Recall that $\ker(T) = \{v \in V \mid T(v) = 0\}$.]

b. Prove that if A and B are linear transformations from V to itself then $\dim\ker(AB) \leq \dim\ker(A) + \dim\ker(B)$.

c. Give an example of linear transformations A and B such that the inequality in part c is strict.

6. Let A be the 4×4 real matrix

$$\begin{bmatrix} 0 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

a. Find the characteristic equation of A and the eigenvalues of A.

b. Find a basis for each eigenspace of A.

c. Is A diagonalizable? If it is, diagonalize it. If not, explain why not.