Masters' Comprehensive Exam
Real and Complex Analysis

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Instructions:

1. If you think that a problem is incorrectly stated ask the proctor. If his or her explanation is not to your satisfaction, interpret the problem as you see fit, but not so that the answer is trivial.

2. From each part solve 4 of the 6 five problems.

3. If you solve more than four problems from each part, indicate the problems that you wish to have graded.

Part A: Real Analysis

1. Let $E \subset \mathbf{R}$ and let $f_n : E \to \mathbf{R}$ be a sequence of functions.

   i. Give the definition of the statement f_n converges uniformly to f on E.

   ii. Determine if the sequence $f_n(x) = \frac{1}{nx+1}$ converges uniformly on the interval $(0, 1)$.

   iii. Prove that if f_n is a sequence of continuous functions that converges uniformly on $[a, b]$ to f, then $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$

2. i. State the Mean Value Theorem.

   ii. Show that if that if $f(x)$ is three times continuously differentiable on $\mathbf{R}$ and there exists points $x_1 < x_2 < x_3 < x_4$ with $f(x_1) = f(x_2) = f(x_3) = f(x_4)$, then there is a point η in the interval (x_1, x_4) with $f^{(3)}(\eta) = 0$.

3. Suppose that $f(x)$ is continuous on $[0, \infty)$ with $f(0) = 0$, and differentiable on $(0, \infty)$ with $0 < f'(x) \le 1$.

i. Show that the function

$$F(x) = 2\int_0^x f(t)dt - (f(x))^2$$

is increasing on $[0, \infty)$

ii. Show that if $x \geq 0$, $2\int_0^x f(t)dt \geq (f(x))^2$

iii. Show that $\left(\int_0^x f(t)dt\right)^2 \geq \int_0^x f(t)^3 dt$

4. Let (M, d) and (X, h) be metric spaces and let $f : M \to X$ be a continuous function. If M is compact, show that f is uniformly continuous on M.

5. Suppose that $f(x) = \sum_{k=1}^{\infty} f_k(x)$ converges uniformly on a set $E \subseteq \mathbf{R}$, and suppose that $g_i(x)$ is bounded sequence that uniformly converges to its limit on E and with $g_i(x) \geq g_{i+1}(x)$ for all i and $x \in E$. Prove that $\sum_{k=0}^{\infty} f_k(x) g_k(x)$ converges uniformly on E.

6. Let (X, d) be a metric space and suppose that $A \subseteq X$ is closed and $K \subseteq X$ is compact. Show that $d(A, K) > 0$ if and only if $A \cap K = \emptyset$ where $d(A, K)$ is the distance between A and K and is defined by

$$d(A, K) = \inf\{d(x, y) | x \in A,\ y \in K\}$$

Part B: Complex Analysis

1. Express in the form $f(z) = u(x, y) + iv(x, y)$ the square root function $f(z) = \sqrt{z}$ with the discontinuity in the negative real axis.

2. Consider the function $f(z) = z + \frac{1}{z}$. Show that the images of circles of radius ρ about $z = 0$ under the transformation defined by $f(z)$ are ellipses, and describe the image of the unit disk.

3. Suppose that $f(z)$ is an analytic function on a domain D and R is a closed and bounded subset of D that is mapped onto the unit disk. Show that if z is an interior point of R, then $f(z)$ lies in the interior of the unit disk.

4. i. Find the power series expansion of for $f(z) = (e^z - 1)^2$ about $z = 0$

ii. Find the Laurent expansion of $f(z) = \dfrac{1}{(z+2)^2}\dfrac{1}{(z-i)^3}$ in the punctured disk, $0 < |z+2| < \sqrt{5}$.

5. Use the inequality $|\int_C f(z)dz| \leq \int_C |f(z)|ds$ (where here ds indicates integration with respect to arc length), to show that if $\mathrm{Log}(z)$ is the principle value of the logarithm function, then for $|z-1| < \rho < 1$, $|\mathrm{Log}(z)| < \dfrac{\rho}{\rho - 1}$

6. Prove that $\displaystyle\int_0^\infty \frac{x\sin(x)}{x^2+1} = \frac{\pi}{2e}$.