MS COMPREHENSIVE EXAM
DIFFERENTIAL EQUATIONS
SPRING 2008

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This exam has two parts, (A) ordinary differential equations and (B) partial differential equations. Do any three of the four problems in each part. Clearly indicate which three problems are to be graded.

Part A: Ordinary Differential Equations

1. Consider the initial value problem

$$u'' - (1 - u^2)u' + u = 0,\ u(0) = 2,\ u'(0) = 3.$$

   (a) Convert the initial value problem given above to a first order system of ordinary differential equations with initial conditions.

   (b) Apply one step of Euler's numerical method for vectors with stepsize $h = .01$ to the first order system with initial conditions found in part (a) above.

   (c) Use the results of part (b) above to estimate the values of $u(.01)$ and $u'(.01)$ in the original initial value problem.

2. (a) Find at least two different solutions to the initial value problem

$$\frac{dy}{dt} = y^{\frac{1}{3}},\ t \geq 0,\ y(0) = 0.$$

   (b) Let $f = f(t, y)$ be defined for $a \leq t \leq b$, $c \leq y \leq d$, with $\frac{\partial f}{\partial y}$ continuous for $a \leq t \leq b$, $c \leq y \leq d$.

   Show that $f = f(t, y)$ is Lipschitz in y, that is, show there is a positive constant K such that $|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|$ where (t, y_1) and (t, y_2) are any two points satisfying $a \leq t \leq b$, $c \leq y_1 \leq d$, $c \leq y_2 \leq d$.

3. Let $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.

   (a) Find all eigenvalues of A.

   (b) For each eigenvalue of A, find all corresponding eigenvectors.

   (c) Find the general solution to $x' = Ax$.

4. (a) Find all critical points of

$$\begin{cases} x' = -x + y - x(y - x) \\ y' = -x - y + 2x^2 y. \end{cases}$$

   (b) Classify the critical point $x = 0$, $y = 0$ as to the type and stability. Refer to attached table 9.3.1. Provide a phase plane portrait.

Part B: Partial Differential Equations

1. Given the equation $u_{xx} + a^2 u_{yy} = 0$ for a constant. Solve the equation by separation of variables and find solutions $u(x, y)$ for each of the cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$ for the separation constant λ.

2. (a) Solve the equation $\frac{\partial u}{\partial x} = xy$ for $u(x, y)$.

   (b) Show that $f(u - x, u - y) = 0$ satisfies the equation $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - 1 = 0$.

3. (a) Solve the equation $yu_{xy} + u_x = 2x$.

   (b) Given the equation $u_{tt} + 4u_{xt} + 4u_{xx} = f(x, t)$ classify it as to <u>type of equation</u>.

4. (a) Given $f(x)$ and $g(x)$ both periodic with period T; prove that all functions of period T form a vector space.

   (b) Find the Fourier series for $f(x) = x(-\pi < x < \pi)$ assumed to be periodic of period 2π. Sketch $f(x)$.