Department of Mathematics
The University of Toledo

Master of Science Degree
Comprehensive Examination
Applied Statistics—In-Class Portion

April 19, 2008

Instructions

Do all problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
Books, notes, and calculators may be used.
This is a three hour test.

1. Suppose $\mathbf{X} = (X_1, X_2, X_3, X_4)' \sim N_4(\mu, \Sigma)$, where

$$\mu = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}. \qquad \Sigma = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

(a) Find the distribution of X_2.

(b) Find the distribution of $X_1 + X_4$.

(c) Find the distribution of $X_2 + X_3 + X_4$.

(d) Find the joint distribution of $(X_1, X_2)'$.

(e) Find the joint distribution of $(X_3, X_4)'$.

(f) Find the conditional distribution of $(X_1, X_2)'$ given $(X_3, X_4)'$.

2. Suppose that the random vector $\mathbf{X} = (X_1, X_2, X_3, X_4)'$ has mean vector and covariance matrix

$$\mu = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}. \qquad \Sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

Let $Y_1 = X_1 + X_2 + X_3 + X_4$, $Y_2 = X_1 + X_2 - X_3 - X_4$, $Y_3 = X_1 - X_2 - X_3 - X_4$, and $Y_4 = X_1 - X_2 + X_3 - X_4$.

(a) Find $E(Y_1)$.

(b) Find $E(Y_2)$.

(c) Find $E(Y_3)$.

(d) Find $E(Y_4)$.

(e) Find $\mathrm{Cov}(X_1, X_2)$.

(f) Find $\mathrm{Cov}(X_1, X_3)$.

(g) Find $\mathrm{Cov}(X_1, X_4)$.

(h) Find $\mathrm{Cov}(X_2, X_3)$.

(i) Find $\mathrm{Cov}(X_2, X_4)$.

(j) Find $\mathrm{Cov}(X_3, X_4)$.

1. (50 points) Here are 5 observations of two regressors x_1, x_2 and dependent variable y.

i	x_{i1}	x_{i2}	y_i
1	-4	7	1
2	-1	5	4
3	0	1	3
4	1	-1	9
5	2	-5	10

Consider the following two linear regression models where ϵ_i's are iid with a normal distribution, $\mathrm{E}(\epsilon) = 0$ and $\mathrm{Var}(\epsilon) = \sigma^2$.

Full Model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i \tag{0.1}$$

Reduced Model

$$y_i = \beta_0 + \beta_1 x_{i1} + \epsilon_i \tag{0.2}$$

Questions about the full model (0.1):

a. What is the design matrix $\boldsymbol{X}$?
b. What is the $\boldsymbol{X_c}$ in the centered design matrix?
c. Calculate $\hat{\boldsymbol{\beta}} = (\hat{\beta}_1, \hat{\beta}_2)$ with the matrix $\boldsymbol{X_c}$.
d. Test the hypotheses:

$$\beta_1 = 0 \quad \text{vs} \quad \beta_1 \neq 0.$$

Use the t-table attached and state your conclusion. $\alpha = 0.05$.

Questions about the reduced model (0.2):

a. Calculate $\hat{\beta}_1$ in the reduced model.
b. Construct the ANOVA table.
c. Predict $\hat{y}_0$ at $x_1 = 5$.
d. Construct a 95% prediction interval for $\hat{y}_0$ above. Is this interval wider than a 95% interval for $\mathrm{E}(y_0)$? Briefly explain why.
e. Calculate the residuals, the studentized residuals, and the studentized deleted residuals.
f. Calculate R^2 and adjusted R_a^2. Which one is a better measure for model selection? Briefly explain why.

TABLE A-3 *t* Distribution: Critical *t* Values

	Area in One Tail				
	0.005	0.01	0.025	0.05	0.10
Degrees of Freedom	Area in Two Tails				
	0.01	0.02	0.05	0.10	0.20
1	63.657	31.821	12.706	6.314	3.078
2	9.925	6.965	4.303	2.920	1.886
3	5.841	4.541	3.182	2.353	1.638
4	4.604	3.747	2.776	2.132	1.533
5	4.032	3.365	2.571	2.015	1.476
6	3.707	3.143	2.447	1.943	1.440
7	3.499	2.998	2.365	1.895	1.415
8	3.355	2.896	2.306	1.860	1.397
9	3.250	2.821	2.262	1.833	1.383
10	3.169	2.764	2.228	1.812	1.372
11	3.106	2.718	2.201	1.796	1.363
12	3.055	2.681	2.179	1.782	1.356
13	3.012	2.650	2.160	1.771	1.350
14	2.977	2.624	2.145	1.761	1.345
15	2.947	2.602	2.131	1.753	1.341
16	2.921	2.583	2.120	1.746	1.337
17	2.898	2.567	2.110	1.740	1.333
18	2.878	2.552	2.101	1.734	1.330
19	2.861	2.539	2.093	1.729	1.328
20	2.845	2.528	2.086	1.725	1.325
21	2.831	2.518	2.080	1.721	1.323
22	2.819	2.508	2.074	1.717	1.321
23	2.807	2.500	2.069	1.714	1.319
24	2.797	2.492	2.064	1.711	1.318
25	2.787	2.485	2.060	1.708	1.316
26	2.779	2.479	2.056	1.706	1.315
27	2.771	2.473	2.052	1.703	1.314
28	2.763	2.467	2.048	1.701	1.313
29	2.756	2.462	2.045	1.699	1.311
30	2.750	2.457	2.042	1.697	1.310
31	2.744	2.453	2.040	1.696	1.309
32	2.738	2.449	2.037	1.694	1.309
34	2.728	2.441	2.032	1.691	1.307
36	2.719	2.434	2.028	1.688	1.306
38	2.712	2.429	2.024	1.686	1.304
40	2.704	2.423	2.021	1.684	1.303
45	2.690	2.412	2.014	1.679	1.301
50	2.678	2.403	2.009	1.676	1.299
55	2.668	2.396	2.004	1.673	1.297
60	2.660	2.390	2.000	1.671	1.296
65	2.654	2.385	1.997	1.669	1.295
70	2.648	2.381	1.994	1.667	1.294
75	2.643	2.377	1.992	1.665	1.293
80	2.639	2.374	1.990	1.664	1.292
90	2.632	2.368	1.987	1.662	1.291
100	2.626	2.364	1.984	1.660	1.290
200	2.601	2.345	1.972	1.653	1.286
300	2.592	2.339	1.968	1.650	1.284
400	2.588	2.336	1.966	1.649	1.284
500	2.586	2.334	1.965	1.648	1.283
750	2.582	2.331	1.963	1.647	1.283
1000	2.581	2.330	1.962	1.646	1.282
2000	2.578	2.328	1.961	1.646	1.282
Large	2.576	2.326	1.960	1.645	1.282

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