Probability and Statistical Theory

MS Comprehensive Examination

April 12, 2008

Instructions:

Please answer all three questions.

Record your answers in your blue books.

Show all of your computations.
Prove all of your assertions or quote the appropriate theorems.
Books, notes, and calculators *may be used.*

You have three hours.

1. Let Z be a standard normal random variable. Let $X = Z, Y = Z^2$.
1). Show that $Cov(X, Y) = 0$.
2). Show that X and Y are not independent.
2. The joint distribution of X and Y is given by the following density
$f(x, y) = ce^{y/\beta}, 0 < x < y < \infty$
where c is a constant and $\beta > 0$.
1). Find the proper value of c that makes this a probability density function.
2). Find the joint distribution of $U = X$ and $V = Y - X$. Are U and V independent?
3). Let $W_1, \cdots, W_n$ be a sequence of iid random variables with the exponential density $f(w) = \frac{1}{\beta}e^{-w/\beta}$ for $0 < w < \infty$. Let $W_{\{1\}} = min(W_1, \cdots, W_n)$. Find the distribution of $nW_{\{1\}}$

3. (50 points) Suppose X_i's ($1 \le i \le n$) are iid with the density function

$$f(x|b) = \frac{x}{b^2} \exp\left(-\frac{x^2}{2b^2}\right), \quad x > 0, b > 0.$$

Answer the following questions:

a. Find $\mathrm{E}(X)$.

b. Show that

$$G = \sum X_i^2 \sim Gamma(n, 2b^2). \tag{0.1}$$

c. Find MLE $\hat{b}$.

d. Find a complete and sufficient statistic for b.

e. Find the UMVUE $\tilde{b}$.

f. Calculate the variance of $\tilde{b}$.

g. Calculate the Cramér-Rao Lower Bound. Does UMVUE reach it?

h. Use the K-R Theorem to find a UMP level α test $\phi_1(\vec{x})$ for

$$H_0 : b = b_0 \text{ vs } H_1 : b > b_0.$$

i. By the test in (h), find a UMA $1-\alpha$ confidence bound for b.

j. Consider $H_0 : b = b_0$ vs. $H_1 : b \neq b_0$. Show that an LRT $\phi_2(\vec{x})$ with the rejection region

$$R = \left\{\vec{x} : \frac{G}{2b_0^2} \le Gamma_{1-\frac{\alpha}{2}}(n, 1) \text{ or } \frac{G}{2b_0^2} \ge Gamma_{\frac{\alpha}{2}}(n, 1)\right\}$$

is a level $\alpha \in (0, 1)$ test, where G is defined in (0.1).