MA exam: algebra, 18 April 2009

Please do four problems, including one from each of the three sections. Give complete proofs — do not just quote a theorem. Please indicate clearly which four problems you want to be graded.

Part I: Group theory

1. Suppose that G is a finite group that has two distinct subgroups of the same order. Prove that G is not cyclic.

2. Let $n \geq 3$ and let H be the subgroup of the symmetric group S_n generated by the set of 3-cycles. Show that H is A_n, the alternating group on n letters.

Part II: Ring theory

3. Consider the polynomial ring $\mathbb{Z}[x]$.
   (a) Prove that $\mathbb{Z}[x]$ is an integral domain.
   (b) Prove that $\mathbb{Z}[x]$ is not a principal ideal domain.

4. Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let R be the set of 2×2 matrices with rational entries that commute with A.
   (a) Prove that R is a ring.
   (b) Prove that $R \cong \mathbb{Q}[x]/I$, where I is the ideal generated by $x^2 - 1$.

Part III: Linear algebra

5. Let V be the vector space of real valued functions having basis $B = \{1, x, e^x, xe^x\}$. Let T be the linear operator on V defined by
$$T(f) = f - \frac{df}{dx}.$$
   Find
   (a) the matrix of T with respect to the basis B,
   (b) the characteristic and minimal polynomials of T,
   (c) the rank and nullity of T,
   (d) bases for the range and nullspace of T,
   (e) all eigenvalues of T.

6. Consider the 4×4 real matrix
$$A = \frac{1}{40} \begin{pmatrix} 40 & -1 & -5 & 2 \\ -1 & 40 & -2 & 5 \\ -5 & -2 & 40 & 1 \\ 2 & 5 & 1 & 40 \end{pmatrix}$$
   Prove that $A^{1001} \neq I$.