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M.S. and M.A. Comprehensive Analysis Exam

Real and Complex Analysis

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To obtain full credit you must show all your work

Part 1. Real Analysis

100% will be obtained for *complete* answers to four questions. Indicate clearly which four questions you wish to be graded.

1. Let $x_1 = 0$ and $x_{n+1} = \dfrac{x_n + 1}{2}$ for $n \geq 1$.

(a) Use induction to show that $x_n < 1$ for all n.

(b) Show that the sequence $\{x_n\}$ is increasing.

(c) Find the limit of the sequence.

2. Let $f_n(x) = \dfrac{1}{nx+1}$, $x \geq 0$.

(a) Find the pointwise limit of the sequence.

(b) Show that if $a > 0$, then the convergence of the sequence is uniform on the interval $[a, +\infty)$.

(c) Show that the convergence is not uniform on the interval $[0, +\infty)$.

3. (a) State the definition: $f(x)$ is Riemann integrable on $[a, b]$.

(b) Let $f(x) = \begin{cases} 1, & \text{if } x \in [1,2]; \\ 0, & \text{if } x \in [0,1). \end{cases}$ Use the definition to prove that f is integrable on $[0, 2]$.

(c) What is $\displaystyle\int_0^2 f(x)\, dx$?

4. Let (M, d) be a metric space and let $B \subset M$.

(a) Show that the closure of $B = \{x | \text{ dist } (x, B) = 0\}$.

(b) Show that the diameter of $B =$ diameter of $\overline{B}$.

5. (a) Determine if the given series converge. Explain your answer.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{(n+1)!}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^5 + 3n^3 - 1}}$

(b) Find the interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)5^n}(x-2)^n$.

Part 2. Complex Analysis

100% will be obtained for *complete* answers to four questions. Indicate clearly which four questions you wish to be graded.

1. (a) Evaluate completely the complex number $(1+i)^{3+4i}$.

(b) Describe geometrically the set of points $z \in \mathbb{C}$ such that $|z+2a| = |2z+a|$ for $a \in \mathbb{C}$.

2. Use residues to find $\displaystyle\int_0^{2\pi} \frac{d\theta}{k+\sin\theta}$, $k > 1$.

3. Expand $f(z) = \dfrac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for:

(a) $0 < |z-1| < 2$

(b) $0 < |z-3| < 2$

4. Let f and g be two analytic functions defined on the open unit disk $\mathbb{D}$ in the complex plane. Suppose that $f(z) + \overline{g(z)}$ is real for all $z \in \mathbb{D}$. Prove that $f - g$ is constant.

5. (a) Evaluate $\displaystyle\int_C \frac{\cos z}{(z-1)^3(z-5)^2}\,dz$, where C is the circle $|z-4| = 2$.

(b) Investigate the analyticity of $f(z) = r^2\cos^2\theta + ir^2\sin^2\theta$ for $z = re^{i\theta} \neq 0$.