MS COMPREHENSIVE EXAM
DIFFERENTIAL EQUATIONS
SPRING 2009

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This exam has two parts, (A) ordinary differential equations and (B) partial differential equations. Do any three of the four problems in each part. Mark clearly the problems you choose and show the details of your work.

Part A: Ordinary Differential Equations

1. Consider the initial value problem

$$u'' - (1-u^2)u' + u = 0, \; u(0) = 2, \; u'(0) = 0.$$

   (a) Convert the initial value problem given above to a first order system of ordinary differential equations with initial conditions.

   (b) Apply one step of Euler's numerical method for vectors with stepsize $h = .01$ to the first order system with initial conditions found in part (a) above.

   (c) Use the results of part (b) above to estimate the values of $u(.01)$ and $u'(.01)$ in the original initial value problem.

2. Consider the second order equation

$$y'' + p(x)y' + q(x)y = 0$$

   on an open interval I where p and q are continuous.

   (a) State the Bolzano-Weierstrass Theorem.

   (b) Let y be a non-zero solution. Prove that all zeros of y are isolated.

   (c) Let y_1 and y_2 be two linearly independent solutions. Prove that between any two consecutive zeros of y_1 there is exactly one zero of y_2.

Part B: Partial Differential Equations

1. Suppose that $g(t) = \int_0^t f(\tau)\, d\tau$. If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$ respectively, show that

$$G(s) = \frac{1}{s} F(s).$$

2. Find the general solution $u(x, y, z)$ for the equation

$$y u_x - x u_y + y z u_z = x$$

by using the method of characteristics.

3. Given the heat equation

$$u_t = u_{xx},$$

assume $u = f\left[\frac{x}{\sqrt{t}}\right]$ and determine the function $f(x, t)$.

4. Use separation of variables to find three general solutions (depending on the separation constant values) to Laplace's equation:

$$u_{xx} + u_{yy} = 0.$$