Department of Mathematics
The University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

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Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
Books, notes, and calculators may be used.
This is a three hour test.

1. Suppose that X is a random variable with pmf
$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, k = 0, 1, \cdots$.

1). Show that the moment generating function of X is $M(t) = \exp(\lambda(e^t - 1))$.

2). Let $Y = \sum_{i=1}^n X_i$ where $X_i, i = 1, \cdots, n$ are iid according to the pmf given above with $\lambda = 1$.

(a). Find the distribution of Y.

(b). Find the limiting distribution of $(Y - n)/\sqrt{n}$. Be sure to identify any theorem that you use to find the limiting distribution.

3). Conditional on $X = x$, we observe an independent random variable Z where $Z \sim$ binomial (x, p). Find the unconditional mean and variance of Z.

2. The joint density of X and Y is given by the following density
$f(x, y) = \frac{1}{\beta^2}e^{-y/\beta}, 0 < x < y < \infty$ where β is a positive constant.

1). Are X and Y independent? Explain.

2). Find the joint distribution of $U = X$ and $V = Y - X$. Show that U and V are independent.

3). Let $W_1, \cdots W_n$ be a sequence of iid random variables with the exponential density $f(w) = \frac{1}{\beta}e^{-w/\beta}$ for $0 < w < \infty$. If $W_{(1)}$ is the minimum of the first n variables in the sequence, find the distribution of $nW_{(1)}$.

3. Suppose that $X_1, \ldots, X_n$ is a random sample from a Rayleigh population with density function

$$f(x, \theta) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \qquad x > 0,\ \theta > 0,$$

where θ is an unknown parameter.

(a) Find a complete and sufficient statistic for θ.

(b) Let $T(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i^2$, find $\mathrm{E}(T)$ and $\mathrm{Var}(T)$.

(c) Find the method of moments estimator of θ^4.

(d) Find the maximum likelihood estimator of θ^4.

(e) Find the distribution of $\frac{1}{\theta^2} \sum_{i=1}^{n} X_i^2$.

4. (25) Let $f(x) = cx$ for $0<x<\tau$, for $\tau>0$. Further let $X_1,\ldots,X_n$ be a sequence of independent observations from the distribution with density function given by $f(x)$.

a. Find c. It is a function of τ.

b. Find the CDF $F(x)$ for this distribution. Be careful to identify the domain.

c. Find the mean and variance for this distribution.

d. Find the most logical method of moments estimator U for τ based on the sample.

e. Find a sufficient statistic S for τ based on the sample.

f. Find and sketch the likelihood function. Be careful to identify the domain.

g. Show that the maximum likelihood estimator V for τ is the maximum of the observations.

h. Find the CDF of V. Be careful to identify the domain.

i. Find the likelihood ratio test statistic λ for testing H_0: $\tau=1$ vs. H_A: $\tau \neq 1$. Hint: It is a function of V and you must consider two cases.

j. Sketch λ as a function of V.

k. At level of significance $\alpha=.1$ and with $n=3$, find the test, i.e., find the critical region using a statistic whose null distribution (distribution under H_0) we know.

l. In the situation of part k, if the data is .2, .5, .9, perform the test.

m. In the situation of part k, if the data is .7, .9, 1.1, perform the test.

n. Find and sketch the power function for the test derived in part k.