MA exam: algebra, 17 April 2010

Please do four problems, including one from each of the three sections. Give complete proofs — do more than simply quote a theorem. Please indicate clearly which four problems you want to be graded.

Part I: Group theory

1. A group (not necessarily finite) is said to be *simple* if it contains no proper non-trivial normal subgroups.

a. Prove that a simple abelian group is finite and cyclic of prime order.

b. If G and H are groups, G is simple and $f : G \to H$ is a non-trivial group homomorphism, prove that H contains a subgroup isomorphic to G.

2. Let G be a finite group and assume that H and K are subgroups of G such that the order of H and the index of K in G are relatively prime.

a. If K is a normal subgroup of G, prove that $H \leq K$.

b. Show by example that the conclusion of part (a) is not valid if the hypothesis that K is normal in G is omitted.

Part II: Ring theory

3. An element a of a ring is said to be *nilpotent* if $a^k = 0$ for some positive integer k.

a. If p is a prime and m is a fixed positive integer, prove that every element of the ring $\mathbb{Z}/(p^m)$ is either a unit or a nilpotent element.

b. Characterize those integers n for which the ring $\mathbb{Z}/(n)$ has no non-zero nilpotent elements.

4. Let $f(x) = x^4 + 6x + 2 \in \mathbb{Q}[x]$, where $\mathbb{Q}$ is the field of rational numbers, and let $I = (f(x))$ be the ideal of $\mathbb{Q}[x]$ generated by $f(x)$. Let $K = \mathbb{Q}[x]/(f(x))$ and let $\alpha = x + I \in K = \mathbb{Q}[x]/(f(x))$

a. Prove that K is a field that contains a subfield isomorphic to $\mathbb{Q}$ and a root of $f(x)$.

b. Find a polynomial $p(x) \in \mathbb{Q}[x]$ such that $\alpha^{-1} = p(\alpha)$.

Part III: Linear algebra

5. Let V be an n-dimensional vector space over the real numbers $\mathbb{R}$.

a. Prove that every 1-dimensional subspace of V is the intersection of all of the $n-1$-dimensional subspaces that contain it.

b. If $T : V \to V$ is a linear transformation such that $T(W) \subseteq W$ for every $n-1$-dimensional subspace W of V, prove that T is a scalar transformation (i.e. prove that $T = cI$ for some $c \in \mathbb{R}$, where I is the identity transformation).

6. Let A be a 3×3 matrix over the field $\mathbb{R}$ of real numbers and suppose that $\mathrm{tr}(A) = 6$, $\mathrm{tr}(A^2) = 14$ and $\det(A) = 6$. Prove that A is similar to the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$