M.S and M.A Comprehensive Analysis Exam
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To get full credit you must show all your work.

Real Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. (a) Define the convergence of a sequence of real numbers.

   (b) Give an example of a convergent sequence $\{x_n\}$ of positive numbers such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$.

   (c) Suppose that $\{x_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L > 1$. Does $\{x_n\}$ converge?

2. Let $\{f_n\}$ be a sequence of real-valued functions on $\mathbb{R}$.

   (a) Define the pointwise convergence of $\{f_n\}$ using the ε-definition.

   (b) Define the uniform convergence of $\{f_n\}$ using the ε-definition.

   (c) For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, let $f_n(x) = \frac{e^{nx}}{e^{nx}+e^{-nx}}$. Show that the sequence $\{f_n\}$ converges pointwise on $\mathbb{R}$ and find its limit f.

   (d) Does $\{f_n\}$ converge to f uniformly on the closed interval $[0,1]$?

3. (a) Let $a_n, b_n \geq 0$ for all n and $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n < \infty$. Prove that the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

   (b) Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)^n (x+1)^n}{n+3}$

4. Let (X, ρ) be a metric space and let A be a nonempty subset of X. Define

$$f(x) = \text{dist}(x, A) = \inf\{\rho(x, y) : y \in A\}.$$

Prove that f is uniformly continuous on X.

5. Let $f(x) = \begin{cases} x, & -1 < x \leq 0 \\ x^2, & 0 < x \leq 1 \end{cases}$. Use the definition of Riemann integrability to prove that f is integrable on $[-1, 1]$.

6. Let $f : (X, \rho_1) \to (Y, \rho_2)$ be a continuous function between two metric spaces (X, ρ_1) and (Y, ρ_2). Assume that K is a nonempty compact subset X. Prove that $f(K)$ is compact.

Complex Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. Discuss and calculate all possible values of the integral $\oint_C \frac{e^z}{z(z-1)^2} dz$ where C is a positively oriented simple closed curve that does not pass through 0 or 1.

2. Find an analytic mapping that maps the domain $\{z \in \mathbb{C} : 1 < \text{Im}(z) < 2\}$ onto the upper half plane.

3. Find an entire function whose imaginary part is $v(x, y) = x^2 - y^2 + 2$.

4. Expand $f(z) = \frac{z+2}{z^2-z-2}$ in a Laurent series valid for

   (a) $2 < |z| < \infty$

   (b) $1 < |z| < 2$

5. Evaluate the integral $\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}$ where $-1 < a < 1$.

6. (a) State Maximum Modulus Principle for analytic functions.

   (b) Let Ω be a domain in $\mathbb{C}$ and u be the real part of an analytic function f on Ω. Assume that u is constant on the boundary of Ω. Show that f is constant on Ω. (Hint: use Maximum Modulus Principle.)