MS COMPREHENSIVE EXAM
DIFFERENTIAL EQUATIONS
SPRING 2010

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This exam has two parts, (A) ordinary differential equations and (B) partial differential equations. Do any three of the five problems from each part. Clearly indicate which three are to be graded and show the details of your work.

Part A: Ordinary Differential Equations

1. Let $A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$.

   (a) Find all eigenvalues of A.

   (b) For each eigenvalue of A, find one corresponding eigenvector.

   (c) Find the general solution of $x' = Ax$.

2. (a) Find all critical points of $\begin{cases} x' = x + y^2 \\ y' = x^2 - y \end{cases}$.

   (b) Classify the critical point $x = -1$, $y = 1$ as to the type and stability. Refer to the attached table. Provide a phase plane portrait showing the direction of increasing t.

3. Recall that the Laplace transform $\mathcal{L}$ is defined by

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)dt.$$

Also recall that

$$\mathcal{L}(y')(s) = s[\mathcal{L}(y)(s)] - y(0).$$

Use Laplace transforms to solve the initial value problem

$$y'' - y' - 2y = 0$$

$$y(0) = 1, \quad y'(0) = 0.$$

4. Consider the second order linear homogeneous ordinary differential equation

$$(1 - x^2)y'' - 2xy' + 6y = 0.$$

Assume a solution of the form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

to find two linearly independent solutions.

5. Let y_1 and y_2 be two differentiable functions defined on (a, b). Suppose that the Wronskian

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

is nonzero where t_0 is in (a, b).

Show that y_1 and y_2 are linearly independent on interval (a, b) by showing that

$$c_1 y_1(t) + c_2 y_2(t) = 0 \text{ for all } t \text{ in } (a, b)$$

implies $c_1 = c_2 = 0$.

Part B: Partial Differential Equations

1. Find the Laplace transform of $f(t) = (t+b)^2$, where b is constant, by using the definition.

2. Show that the function $z = f[x + g(y)]$ satisfies the PDE: $z_{xx}z_y = z_{xy}z_x$, for f and g are arbitrary functions.

3. Given $z^3(x,y) + 3xyz(x,y) + a^3 = 0$ for a constant. Prove that $x^2 z_{xx} = y^2 z_{yy}$.

4. Find the general solution $u(x,y)$ to equations $u_{xy} = 1$ and $u_{yx} = 1$ respectively. Compare the solutions and make a conclusion about the mixed derivatives.

5. Find the general solution $u(x,y,z)$ to the equation

$$yu_x - xu_y + yzu_z = x$$

by using the <u>method of characteristics</u>.