Department of Mathematics
The University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

April 10, 2010

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
Books, notes, and calculators may be used.
This is a three hour test.

1. A deck of 52 cards is shuffled and split into two halves. Let X be the number of red cards in the first half.

(a) Find the probability frequency distribution of X.

(b) Find $E(X)$.

(c) Find $\text{Var}(X)$.

2. Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy

$$Y_i = \beta x_i + \epsilon_i, \qquad i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are fixed constants, and $\epsilon_1, \ldots, \epsilon_n$ are iid $N(0, \sigma^2)$ with σ^2 unknown.

(a) Find a two-dimensional sufficient statistic for (β, σ^2).

(b) Find the maximum likelihood estimator $\hat{\beta}$ of β. Is $\hat{\beta}$ an unbiased estimator of β?

(c) Assume that we use $\tilde{\beta} = \sum_{i=1}^n Y_i / \sum_{i=1}^n x_i$ to estimate β. Is $\tilde{\beta}$ an unbiased estimator of β?

(d) Calculate the exact variances of $\hat{\beta}$ and $\tilde{\beta}$ and compare them.

(e) Find the distributions of $\hat{\beta}$ and $\tilde{\beta}$.

3. (25 points) The joint probability density function of X and Y is given by

$$f(x,y) = c\left(x^2 + \frac{xy}{2}\right) \qquad 0 < x < 1, 0 < y < 2$$

a. Find c.
b. Find the density function $f_X(x)$ of X.
c. Find $\mathrm{P}(X > Y)$.
d. Find $\mathrm{P}(Y > 1/2 | X < 1/2)$.
e. Find $\mathrm{E}(X)$.

4. (25 points) Suppose $X_1, \cdots, X_n \overset{iid}{\sim} f(x|\theta) = \dfrac{\theta}{x^2} \qquad (x > \theta > 0)$.

a. Find a sufficient statistic.
b. Find the distribution of $Y = X_{(1)}$.
c. Use the Neyman-Pearson Lemma to show that the test $\phi(y)$

$$\phi(y) = \begin{cases} 1, & y > \theta_0 \alpha^{-1/n} \\ 0, & \text{otherwise,} \end{cases}$$

is a UMP level α test for

$$H_0 : \theta = \theta_0 \text{ vs. } H_1 : \theta = \theta_1 \qquad (\theta_1 > \theta_0 > 0).$$

d. Find MLE for θ $(\theta > 0)$.
e. Show that the level α LRT for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$ $(\theta_0 > 0)$ has the rejection region:

$$R = \{y : y < \theta_0 \text{ or } y > \theta_0 \alpha^{-1/n}\}.$$