Masters Comprehensive Exam
Real and Complex Analysis
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To obtain full credit you must show all your work

Part 1. Real Analysis. Answer 4 of the 6 questions in Part 1. If you answer more questions then indicate which you wish to be considered.

(1) Suppose that f is Riemann integrable on $[a, b]$ and $g = f$ except at one point $c \in [a, b]$. Prove that g is Riemann integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$$

(2) (a) Define continuity for $f : [a, b] \to \mathbb{R}$.
(b) Suppose that f, g are continuous on $[a, b]$ and define

$$\begin{aligned} h(x) &= \min\{f(x), g(x)\} \\ H(x) &= \max\{f(x), g(x)\} \end{aligned}$$

for all $x \in [a, b]$. Show that h and H are continuous on $[a, b]$.

(3) (a) Define the convergence of a sequence in a metric space (X, d).
(b) Show that every convergent sequence of real numbers has a monotone subsequence that converges to the same limit.

(4) Suppose that $f : [1, \infty) \to \mathbb{R}$ is a continuous function and $\lim_{x\to\infty} f(x) = A$. Show that

$$\lim_{t\to\infty} \frac{1}{t} \int_1^t f(x)\,dx = A$$

(5) Define the function

$$f(x) = \frac{2x}{1 + x^2}$$

and let $f_n(x) = f(nx)$.
(a) Show that the sequence f_n converges to 0 uniformly on the interval $[1, \infty)$
(b) Does the sequence f_n converge to 0 on $[0, 1]$? Is the convergence uniform?

(6) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f(0) = 0$, and $f'(x) > f(x)$ for all $x \in \mathbb{R}$. Prove that $f(x) > 0$ for $x > 0$.

Part 2. Complex Analysis. Answer 4 of the 5 questions in Part 2. If you answer more questions then indicate which you wish to be considered.

(1) (a) The symbol i^i is infinitely many valued. Find these values. They are all real positive.
(b) Solve the transcendental equation $\cos z = \sqrt{3}$

(2) Show that $u(x, y) = 6x^2y + 6xy - 2y^3$ is harmonic. Then find a function $v(x, y)$ so that $f(x + iy) = u(x, y) + iv(x, y)$ is analytic.

(3) Write the Laurent series in powers of z that represents the function

$$f(z) = \frac{z - 1}{z(z^2 + 1)}$$

in certain domains and specify those domains.

(4) (a) Evaluate the integral $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 3\sin\theta}$

(b) By integrating $\frac{e^{iz}}{z}$ over a suitable contour, evaluate

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

(5) Evaluate the contour integrals. Here C is the circle $\{z : |z - i| = 1\}$ oriented counterclockwise.

(a) $\displaystyle\int_C e^{z^2} + z^4 + \frac{z^3 + 4z}{2z - i} + \frac{1}{z^2 + 1}\, dz$

(b) $\displaystyle\int_{|z|=1} \frac{1}{z - \sin z}\, dz$