MS Comprehensive EXAM
DIFFERENTIAL EQUATIONS
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This exam has two parts, ordinary differential equations and partial differential equations. Choose four problems in each part.

Part I: Ordinary Differential Equations

1. Find the solution of the initial value problem on $(0, \infty)$.

$$y''(x) - \frac{3}{x} y'(x) + \frac{20}{x^2} y(x) = 0, \quad y(1) = 1, \quad y'(1) = -2.$$

2. Find three linearly independent solutions of the equation and show that these solutions are linearly independent.

$$y''' - y'' + y' - y = 0.$$

3. Draw the phase portrait of the differential equation on the half plane $x \geq 0$. Find the smallest v_0 such that the solution $x(t)$ of the equation with the initial conditions $x(0) = 0, \dot{x}(0) = v_0$ satisfies $\lim_{t \to \infty} x(t) = \infty$.

$$\ddot{x} = -\frac{12}{(3 + x)^2}$$

4. Compute eigenvalues and eigenfunctions of the following Sturm-Liouville problem.

$$y''(x) + \lambda y(x) = 0; \quad y'(0) = 0, \ y(\pi) = 0.$$

5. Solve the nonhomogeneous linear system for $x \in R^2$ with the initial condition.

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ e^{2t} \end{bmatrix} \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

6. Let $r_1(t) = (x_1(t), y_1(t)), r_2(t) = (x_2(t), y_2(t))$ denote the positions of two point masses with $m_1 = 20, m_2 = 10$. Assume Newton's law of universal gravitation with $G = 1.5$ and Newton's second law of motion. Write a system of differential equations for $x_1(t), y_1(t), x_2(t), y_2(t)$.

7. Consider a homogeneous n-th order linear ODE ($n \geq 1$):

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0,$$

on an interval I of the real line with real or complex valued continuous function p_{n-1}. Let $y_1, \ldots, y_n$ be n real or complex valued solutions of the given n-th order ODE and consider the Wronskian $W(y_1, \ldots, y_n)(x)$, $x \in I$. Show that the Wronskian satisfies the following formula (Abel's formula):

$$W(y_1, \ldots, y_n)(x) = W(y_1, \ldots, y_n)(x_0) \exp\left(-\int_{x_0}^{x} p_{n-1}(\xi)\, d\xi\right), \quad x \in I$$

for every point x_0 in I.

Part II: Partial Differential Equations

1. Find a solution of the initial value problem for the heat equation.

$$\begin{cases} u_t(x,t) = 9u_{xx}(x,t) & \text{on } -\infty < x < \infty,\ t > 0; \\ u(x,0) = e^{-2x^2} + 1 & \text{for } -\infty < x < \infty. \end{cases}$$

2. Find all radially symmetric solutions of $\Delta u + 4u = 0$ in R^3.

3. Solve the initial-boundary value problem for the wave equation.

$$\begin{cases} u_{tt}(x,t) = 9u_{xx}(x,t) & \text{on } 0 \le x \le \pi, t > 0; \\ u(0,t) = 0, \quad u(\pi,t) = 0 & \text{for } t > 0; \\ u(x,0) = \sum_{n=1}^{10} a_n \sin(nx), & \\ u_t(x,0) = 0 & \text{for } 0 \le x \le \pi. \end{cases}$$

4. Let $u(x,y)$ be a function with continuous second derivatives that satisfies the following conditions. Find $u(x,y)$ and explain the result.

$$u_{xy}(x,y) = 0, \quad u(x,0) = x^2, \quad u(0,y) = -y^2.$$

5. For the $u(x,y)$ as below, find $v(x,y)$ satisfying the Cauchy-Riemann equations $\partial u/\partial x = \partial v/\partial y, \partial u/\partial y = -\partial v/\partial x$ and the condition $v(0,0) = 0$.

$$u(x,y) = xy + x^3 - 3xy^2$$

6. Solve the initial value problem.

$$\begin{cases} u_t(x,t) + u(x,t)u_x(x,t) = 0 & \text{on } -\infty < x < \infty,\ t > 0; \\ u(x,0) = 2x - 1 & \text{for } -\infty < x < \infty. \end{cases}$$

7. Consider the following evolutionary PDE on the real line $(-\infty, +\infty)$ for an unknown function $u(x,t)$, which is assumed to be sufficiently differentiable:

$$u_t = uu_x + u_{xxx}.$$

Show that if one is looking for traveling wave solutions $u(x,t) = f(x - ct)$, the given PDE can be reduced to a nonlinear ODE (but do not try to solve the corresponding ODE).