M.A Comprehensive Topology Exam

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To get full credit you must show all your work. This exam contains 10 questions, do 5 out of them. Please indicate clearly which answers you want to be graded.

This examination has been checked carefully for errors. If you find what you believe to be an error in a question, report this to the proctor. If the proctors interpretation still seems unsatisfactory to you, you may modify the question so that in your view it is correctly stated, but not in such a way that it becomes trivial. If you feel that the examination is on the long side do not panic. The grading will be adjusted accordingly. Please write as legibly as you can: the examination may need to be scanned for grading.

1 Do 5 problems

1. (a) Define the term *closure* $\bar{A}$ for a subspace A of a topological space X.

   (b) Prove that if A and B are subspaces of X, $A \subset B$ and B closed then $\bar{A} \subset B$.

2. (a) Define the notion of topological space X.

   (b) Define the notion of discrete topology on X.

   (c) Show that any function $f : X \to Y$, X, Y topological spaces, with X equipped with the discrete topology is necessarily continuous.

3. (a) Define the notion of metric space (X, d).

   (b) Is the metric function $d : X \times X \to \mathbb{R}$ continuous? Prove it or disprove it.

4. (a) Define the notion of compactness for a topological space.

   (b) Let X be a compact topological space, Y topological space and $f : X \to Y$ continuous. Show that $f(X)$ is compact.

5. Prove that a continuous bijection from a compact topological space onto a Hausdorff topological space is necessarily a homeomorphism.

6. Prove or disprove: is the map $f : [0, 2\pi) \to \mathbb{S}^1$ given by $t \mapsto (\cos(t), \sin(t))$ a homeomorphism? Here $\mathbb{S}^1$ is the unit circle in the plane $\mathbb{R}^2$ with its standard topology.

7. (a) Define the product topology for a product of topological spaces $X \times Y$.

   (b) Prove from your definition that each of the projections $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$ are continuous and open maps.

   (c) Show with an example that π_X and π_Y are not closed maps.

8. (a) Define what it means for a topological space to be disconnected.

   (b) Prove that a space is disconnected if and only if there is a continuous map from the space onto the discrete two point space $\{0, 1\}$.

9. (a) Define the notion of a connected subset A of a topological space X.

   (b) Prove that if for two connected sets A and B of a topological space X there exists $p \in A \cap B$, then $A \cup B$ is also connected.

10. Prove that for any given subset E in a metric space (X, d) the function $f : X \to \mathbb{R}$ defined by

$$f(x) = d(x, E), \quad x \in X$$

is continuous (here $\mathbb{R}$ is equipped with the standard topology).