Department of Mathematics
The University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

April 14, 2012

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
Books, notes, and calculators may be used.
This is a three hour test.

3. Suppose $U_1, U_2, \ldots$ are independent uniform $U(0,1)$ random variables, and let N be the first $n \geq 2$ such that $U_n > U_{n-1}$.

(a) Find $P(U_1 \leq u \text{ and } N = n)$ for $0 \leq u \leq 1$ and $n \geq 2$.

(b) Find $P(U_1 \leq u \text{ and } N \text{ is even})$ for $0 \leq u \leq 1$.

(c) Find $E(N)$.

4. Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ where μ is unknown but σ^2 is known with $-\infty < \mu < \infty$ and $0 < \sigma^2 < \infty$. Write $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$.

(a) Find a complete and sufficient statistic for μ.

(b) Find the method of moments estimate of μ.

(c) Find the maximum likelihood estimates of μ and μ^2.

(d) Find the Fisher information about μ contained in $X_1, \ldots, X_n$.

(e) Find the conditional distribution of X_1 given $\bar{X} = t$.

(f) Find the UMVU estimator of μ. Does the UMVU estimator achieve the Cramér-Rao lower bound? Explain your reasoning.

(g) Find the UMVU estimator of μ^2. Does the UMVU estimator achieve the Cramér-Rao lower bound? Explain your reasoning. In addition, comment on whether or not the UMVU estimator is a suitable estimator of μ^2.

(h) Find the UMVU estimator of $g(\mu) = P_\mu(X_1 > a)$, where a is some fixed and known real number. Does the UMVU estimator achieve the Cramér-Rao lower bound? Explain your reasoning.

1. Let $Y_1,\ldots,Y_n$ be Uniformly distributed over the interval $[0,\theta]$.

a. Show that a sufficient statistic U for inference regarding θ is the maximum, $Y_{(n)}$.
b. Find the CDF of U.
c. Find the method of moments estimator for θ based upon this data.
d. Find the maximum likelihood estimator for θ based upon this data.
e. Find the bias, variance, and mean square error for the method of moments estimator.
f. Find the bias and mean square error for the maximum likelihood estimator using the fact that the variance is $n\theta^2 / (n+1)(n+2)^2$ .Compare these two mean square errors. Which estimator is "better"?
g. For testing H_0: $\theta = 1$ versus H_A: $\theta < 1$ at level of significance α, show that the likelihood ratio test statistic is $Y_{(n)}$ and derive the test. Find the critical value c as a function of n and α.
h. Derive and sketch the power function for this test.
i. If $n=10$ and $\alpha = 0.10$, find the critical region (rejection region) and draw a conclusion if we observe the following (ordered) data: .08, .12, .13, .22, .29, .39, .46, .54, .66, .73.
j. For the above data, calculate the two estimators from parts d & e.

2. Let (X,Y) have continuous joint density that is uniform over the triangle with vertices (0,0) , (0,1) and $(\gamma,0)$ where γ is a positive parameter.

a. Find c.
b. Find the marginal density of X.
c. Find E(X).
d. Find the marginal density of Y.
e. For each y in Range(Y), show that the conditional density of X|Y=y is a continuous uniform distribution. Be sure to identify the interval.
f. For each y in Range(Y), find the conditional expectation E(X|Y=y).
g. Find the density function of the random variable E(X|Y).
h. Find the expectation for the distribution you found for E(X|Y) in part g.
i. Does E{E(X|Y)} from part h equal E(X) from part c? It should.
j. Suggest a way to estimate γ from one observation on (X,Y). Justify your answer as "a way", but necessarily the optimal way.