MS COMPREHENSIVE EXAM
DIFFERENTIAL EQUATIONS
SPRING 2013

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This exam has two parts, (A) ordinary differential equations and (B) partial differential equations. Do any three problems in each part. Clearly indicate which three problems are to be graded.

Part A: Ordinary Differential Equations

1. Recall that the Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, defined by:

$$F(s) = \mathfrak{L}(f(t))(s) = \int_0^\infty e^{-st} f(t)dt.$$

   Suppose that $g(t) = \int_0^t f(\tau)d\tau$. Prove that $\mathfrak{L}(g(t)) = \frac{\mathfrak{L}(f(t))}{s}$.

2. Sketch the bifurcation diagrams and describe the different bifurcation that occurs in the following system.

$$x' = (x - a)(x^2 + a^2 - 4)$$

3. The differential equation $xy''(x)+y(x)+\frac{\omega^2}{g}y(x) = 0$ arises in the study of the vibrations of a hanging chain. Change the independent variable, using $x = \frac{gt^2}{4}$, to obtain a new equation in t and write the solution of the new equation.

4. Find the general solution of the following linear system and determine its stability.

$$x' = Ax \quad \text{where } A = \begin{pmatrix} -1 & -1 \\ 2 & -3 \end{pmatrix}.$$

   Note that $det(A - \lambda I) = \lambda^2 + 4\lambda + 5$.

5. Find the general solution of the following linear system and determine its stability.

$$x' = Ax \quad \text{where } A = \begin{pmatrix} -4 & -1 \\ 1 & -2 \end{pmatrix}.$$

   Note that $det(A - \lambda I) = \lambda^2 + 6\lambda + 9$.

6. Prove that the Picard iteration of the differential equation $X' = AX$ with $X(0) = X_0$ converges to $X(t) = exp(At)X_0$. Here A is a $n \times n$ matrix.

Part B: Partial Differential Equations

1. Find the explicit formula for a function u solving the initial value problem
$$\begin{cases} u_t + b \cdot Du - u = 0, & in\ R^n \times (0, \infty) \\ u = g, & on\ R^n \times \{t = 0\}. \end{cases}$$

2. (a) Let $\phi(s) = \frac{1}{n\alpha(n)s^{n-1}} \int_{\partial B(x,s)} u(y) dS(y)$. Prove that
$$\phi'(s) = \frac{1}{n\alpha(n)s^{n-1}} \int_{B(x,s)} \triangle u(y) dy$$
if $u \in C^2(B(x, s))$.

   (b) We say $u \in C^2(U)$ is harmonic if $\triangle u = 0$ in U. Prove that for harmonic function u that
$$u(x) = \frac{1}{\alpha(n)r^n} \int_{B(x,r)} u(y) dy$$
for all $B(x, r) \subset U$.

3. Find the general solution $u(x, y)$ to
$$u_{xyy} = 1.$$

4. Associated with the heat equation $u_t = u_{xx}$, $t > 0$, $0 < x < \pi$, $u(0, t) = u(\pi, t) = 0$ is a characteristic value problem found by letting $u(x, t) = A(x)e^{\lambda t}$. Find its eigenvalues λ and eigenfunctions $A(x)$.