Department of Mathematics
The University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

April 13, 2013

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
Books, notes, and calculators may be used.
This is a three hour test.

1. (20) Let (X,Y) have density f(x,y) = cx over the triangle with vertices (0,0), (1,0), and (0,1).
   a. Show that c must equal 6.
   b. Show that the marginal of X is Beta(α=2,β=2). Find its mean and variance.
   c. Show that the marginal of Y is Beta(α=1,β=3). Find its mean and variance.
   d. Show that the conditional distribution of Y given X=x is U(0,1-x).
   e. Find E(Y|X=x) and Var(Y|X=x).
   f. Find Cov(X,Y) and the correlation ρ between X and Y.
   g. Confirm the formulas for conditional expectation and conditional variance for this example, i.e., that E(E(Y|X)) = E(Y) and Var(E(Y|X)) + E(Var(Y|X)) = Var(Y) for this joint distribution.
2. (30) Let X_i, for i = 1,2,...,n, be independent and identically distributed random variables with common density function $f(x; \theta,\beta) = \frac{1}{\theta} e^{-(x-\beta)/\theta}$ for $x \in [\beta, \infty)$ and parameter space given by $\theta>0$ and $\beta \in (-\infty, \infty)$. Note that this is a shifted exponential, shifted by the amount β.
   a. Show that the moment generating function for this distribution is given by

      $M(t) = e^{\beta t}/(1-\theta t)$. What is the domain of M?
   b. Use the formula in part a to find the mean and variance for this distribution. Compare your answer to what you should get by noting the fact that this is a shifted exponential.
   c. Find the likelihood function for the n observations on this distribution as a function of the parameters.
   d. Find the sufficient statistic(s) for this model.
   e. Find the method of moments estimators for the two parameters.
   f. Find the maximum likelihood estimator of θ under the restriction that β = 0.
   g. Find the maximum likelihood estimators of θ and β with no restrictions other than given by the parameter space.
   h. Consider a test of H_0:β=0 versus H_A: $\beta \neq 0$. Show that the likelihood ratio λ is given by

$$\lambda(x_1,\ldots,x_n) = \begin{cases} \left(1 - x_{\{1\}}/\bar{x}\right)^n & if\ x_{(1)} \geq 0 \\ 0 & if\ x_{(1)} < 0 \end{cases}.$$

   i. Argue that we therefore reject the null hypothesis in favor of the alternative if the ratio $x_{\{1\}}/\bar{x}$ is greater than some constant c which depends on the level of significance α, or if the minimum $x_{(1)}$ is negative.

3. Consider an experiment in which balls numbered $1, \ldots, n$ are distributed at random in n boxes, also numbered $1, \ldots, n$, so that each box has exactly one ball. Thus, the total number of possible outcomes is $n!$. Let S_n be the number of matches; a match occurs when the ball and the box containing it have the same number. Find $\mathrm{E}(S_n)$ and $\mathrm{Var}(S_n)$. Justify your answers.

4. Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ population, where $-\infty < \mu < \infty$ and $\sigma^2 > 0$.

(a) Find a complete and sufficient statistic for (μ, σ^2).

(b) Let $\bar{X} = \frac{1}{n} \sum_{i=1} X_i$ and $S_n^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2$. Find $\mathrm{Var}(\bar{X}^2 + S_n^2)$.

(c) If σ^2 is known, find the maximum likelihood estimator of $\mu(1-\mu)$.

(d) If σ^2 is known, find the UMVU estimator of $\mu(1-\mu)$. Does the UMVU estimator achieve the Cramér-Rao lower bound?

(e) Use the measure of *mean square error* to compare the maximum likelihood estimator in part (c) with the UMVU estimator in part (d). Which estimator is better? Explain your reasoning.