M.S and M.A Comprehensive Analysis Exam

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To get full credit you must show all your work.
This exam contains 6 real analysis and 6 complex variables questions.

Real Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. (a) Define the supremum of a bounded set $A \subset \mathbb{R}$.

   (b) Let A and B be bounded subsets of real numbers. Show that

$$\sup(A \cup B) = \max\{\sup A, \sup B\}.$$

2. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\int_a^b f(x)dx = 0$ for all $0 \le a \le b \le 1$. Show that $f = 0$.

3. (a) Define uniformly continuous functions on $\mathbb{R}$.

   (b) Show that $f : (0,1) \to \mathbb{R}$ defined by $f(x) = \sin(1/x)$ is not uniformly continuous.

   (c) Show that $g : (0,1) \to \mathbb{R}$ defined by $g(x) = x\sin(1/x)$ is uniformly continuous on $(0,1)$.

4. Prove the root test for series.

   (a) Show that if $\lim\limits_{n\to\infty} |a_n|^{1/n} < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

   (b) Give two examples of series $\sum\limits_{n=1}^{\infty} c_n$ and $\sum\limits_{n=1}^{\infty} d_n$ so that

$$\lim_{n\to\infty} |c_n|^{1/n} = \lim_{n\to\infty} |d_n|^{1/n} = 1$$

   yet $\sum\limits_{n=1}^{\infty} c_n$ is convergent while $\sum\limits_{n=1}^{\infty} d_n$ is divergent.

5. Let $f : [0,1] \to [0,1]$ be continuous. Show that there exists $c \in [0,1]$ such that $f(c) = c$.

6. (a) Define compact subset of real numbers.
   (b) Let $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Using the definition of compactness to prove that S is a compact set.
   (c) Is $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ compact? Prove your assertion.

Complex Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. Find the image of line line $y = x$ under the mapping $f(z) = z^2 - 1$.

2. Let C denote the unit circle with counter-clockwise orientation. Compute the integral

$$\oint_C \frac{e^{z^2}}{z^2} dz.$$

3. Expand $f(z) = \dfrac{z^2}{(z+1)(z^2-1)}$ in a Laurent series valid for $\{z \in \mathbb{C} : 0 < |z-1| < 2\}$.

4. Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)}$.

5. Evaluate the line integral $\displaystyle\int_C z^{1/2} dz$ where C is the positively oriented semicircular curve $z = e^{i\theta}$ for $-\pi/2 \le \theta \le \pi/2$ and $z^{1/2}$ is defined with the standard branch cut $|\text{Arg}(z)| < \pi$.

6. Let u and v be real valued functions continuously differentiable functions on a domain Ω. Assume that u is harmonic conjugate of v and v is harmonic conjugate of u on Ω. Show that both u and v are constant functions.