MS COMPREHENSIVE EXAM
DIFFERENTIAL EQUATIONS
SPRING 2014

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This exam has two parts, (A) ordinary differential equations and (B) partial differential equations. Do any three of the six problems in part A and any three of the seven problems in part B. Clearly indicate which three problems in each part are to be graded. Show the details of your work.

Part A: Ordinary Differential Equations

1. Consider a non-autonomous system of ODEs of the form $\dot{x} = Ax + b(t)$, where A is a constant $n \times n$ matrix with real coefficients and $b(t)$ is a continuous vector valued function in $\mathbb{R}^n$, and $x \in \mathbb{R}^n$.

   (a) Write down a formula to solve the IVP $\dot{x} = Ax + b(t)$, $x(0) = x_0$ and prove the formula.

   (b) Use the formula to solve the IVP for the following equation (here $x \in \mathbb{R}$): $\dot{x} = x + \sin(e^{-t})$.

2. Let $A = \begin{pmatrix} 1 & 1 \\ 9 & 1 \end{pmatrix}$.

   (a) Find all eigenvalues of A.

   (b) For each eigenvalue of A, find the corresponding eigenvectors.

   (c) Draw the phase portrait of $\dot{x} = Ax$ nearby the origin in $\mathbb{R}^2$.

   (d) Find the general solution to $\dot{x} = Ax$.

3. Consider the following second order differential equation $\frac{d^2}{dt^2}x + \frac{\partial}{\partial x}V(x) = 0$, where $x \in \mathbb{R}$ and $V(x) = x^3 - 3x^2$.

(a) Show that the function $E(x, \dot{x}) = \frac{1}{2}(\dot{x})^2 + V(x)$ is conserved along the solutions of the differential equation.

(b) Draw the phase portrait of the equation in the plane $(x, \dot{x})$.

(c) Suppose that at time $t_0 = 0$, $x(0) = 1$, $\dot{x}(0) < 0$ and the corresponding value of $E = 0$. Determine $\dot{x}(0)$ and write down in integral form the time T it takes for the solution to travel from this initial condition to the point $(x, \dot{x}) = (0, 0)$ in the phase plane.

(d) Show that this time is infinite (either estimating the improper integral or invoking a suitable result from the theory of ODEs).

4. (a) For the system $\dot{x} = xy$, $\dot{y} = x + y$ locate its fixed point (or critical point or equilibrium) and calculate its index.

   (b) Study the stability of the origin $(0, 0) \in \mathbb{R}^2$ for the system $\dot{x} = -x + 4y$, $\dot{y} = -x - y^3$: use the strict Lyapunov function $V(x, y) = x^2 + ay^2$, where a is a real parameter to be chosen so that V is indeed a strict Lyapunov function.

5. Show that the system in $\mathbb{R}^3$ $\dot{x} = x\cos^4(t) - z\sin(2t)$, $\dot{y} = x\sin(4t) + y\sin(t) - 4z$, $\dot{z} = -x\sin(5t) - z\cos(t)$ has at least one unbounded solution. (Use Floquet theory.)

6. Consider the following system in the plane minus the origin written in polar coordinates (r, θ): $\dot{r} = 1 - r(2 + \cos(\theta))$, $\dot{\theta} = 1$.

   (a) Show that there exists an annulus $D = \{(r, \theta) \in \mathbb{R}^2 \setminus \{0\} | 0 < r_1 < r < r_2\}$ which is (forward) invariant for the system, determining suitable constants r_1 and r_2 such that $0 < r_1 < r_2$. This means that if a solution starts at time t_0 in D it will remain in D for all $t \geq t_0$. [Hint: Look at the signs of $\dot{r}$ at the boundaries of D...].

   (b) Conclude that there is a periodic orbit inside D invoking a suitable result from the theory of ODEs.

Work completely any three of the seven problems.

Part B: Partial Differential Equations

1. Let $(x, y) \in \mathbb{R}^2$, let $\Sigma := \{(x, 0)\}$ and consider the following IVP for a linear PDE:

$$(1 + x^2)u_x + (1 + y^2)u_y = 0 \text{ for } (x, y) \in \mathbb{R} \times (0, +\infty), \quad u(x, 0) = u_0(x),$$

where u_0 is a smooth function of x.

   (a) Show that Σ is noncharacteristic.

   (b) Find the solution of the IVP above using the method of characteristics.

   (c) Determine if the solution found with the method of characteristics is globally defined or not in $\mathbb{R} \times [0, +\infty)$

2. Solve the following Initial/Boundary value problem for the heat equation in one dimension using separation of variables (since the initial value u_0 is unspecified, just write the formula for the Fourier coefficients). Here u_0 is a given continuous function on $[0, \pi]$.

$$u_t - u_{xx} = 0 \text{ in } (0, \pi) \times (0, +\infty)$$
$$u(x, 0) = u_0(x), \ x \in (0, \pi)$$
$$u(0, t) = u(\pi, t) = 0, \ \forall t \in (0, +\infty).$$

3. Let Ω be a bounded domain in $\mathbb{R}^n$ and $u \in C^0(\bar{\Omega})$ be harmonic in Ω. Then prove that u attains its maximum only on $\partial\Omega$ unless u is constant. [Hint: use the mean value property].

4. Let Ω be a domain in $\mathbb{R}^n$. For any ball $B_r(x) \subset \Omega$, for any $\rho \in (0, r)$ and for any $u \in C^2(\Omega)$ we have that

$$\int_{B_\rho(x)} \Delta u dx = \rho^{n-1} \frac{\partial}{\partial \rho} \int_{\partial B_1} u(x + \rho w) dS_w$$

where B_1 is the open ball centered at zero of radius one. Use the identity above to show that if $u \in C^2(\Omega)$ is harmonic in Ω then it satisfies the mean value property.

5. Find the Green function for the Laplace operator in the first quadrant in $\mathbb{R}^2$.

6. Let Ω be a bounded domain in $\mathbb{R}^n$. Define $\Omega_T := \Omega \times (0, T]$ and the parabolic boundary of Ω_T, defined as $\partial_P \Omega_T := \bar{\Omega}_T \setminus \Omega_T$. Suppose $u \in C^{2,1}(\Omega_T) \cap C^0(\bar{\Omega}_T)$ satisfies

$$u_t - \Delta u \leq 0 \text{ in } \Omega_T.$$

Prove that

$$\max_{\bar{\Omega}_T} u = \max_{\partial_P \Omega_T} u.$$

[Hint: first assume $u_t - \Delta u < 0$ and prove it by contradiction. Next reduce the case $u_t - \Delta u < 0$ setting $u_\epsilon(x,t) = u(x,t) - \epsilon t$ for each $\epsilon > 0$.]

7. Consider the IVP for the one-dimensional wave equation:

$$u_{tt} - u_{xx} = 0 \text{ for } (x,t) \in \mathbb{R} \times (0, +\infty)$$

$$u(x,0) = \phi(x), \; u_t(x,0) = \psi(x) \text{ on } \mathbb{R}$$

where $\phi \in C^2(\mathbb{R})$, $\psi \in C^1(\mathbb{R})$.

(a) Prove d' Alembert's formula.

(b) Use d' Alembert formula to solve the IVP above with $\phi(x) = \sin(x)$ and $\psi(x) = 0$ and explain what is the solution in terms of traveling waves.