Department of Mathematics
The University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

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Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
Books, notes, and calculators may be used.
This is a three hour test.

1. The goal of this problem is to explore a simple fact. If X is Poisson with the parameter (mean) not fixed but exponentially distributed, then X has a geometric distribution. In several steps, we will establish and explore this. Let Y ~ exponential(θ) and X | Y=y ~ Poisson(y). *Here are some helpful facts (you don't need to prove them)* ...
   - Y ~ exponential(θ) means that $f(y;\theta) = {}^{1}/_{\theta}\, e^{-y/\theta}$ for y>0 and parameter space $\theta > 0$. Also E(Y) = θ and Var(Y) = θ^2.
   - X ~ Poisson(τ) means that $p(x;\tau) = {}^{e^{-\tau}\tau^x}/_{x!}$ for x=0,1,2,... and parameter space $\tau > 0$. Also E(X) = τ and Var(X) = τ.
   - W ~ geometric(p) means that $p_W(w) = p(1-p)^w$ for w=0,1,2,... , E(W) = (1-p)/p and Var(W) = (1-p)/p^2. Note that here W counts the number of failures until the first success (it starts at 0).
   - It will be helpful to understand that for the indicator random variable I_A, conditional expectations can be understood as conditional probabilities: just like $E(I_A) = P(A)$, also $E(I_A|B) = P(A|B)$ for any event B with P(B)>0 and $E(I_A|V) = P(A|V)$ for any random variable V.
   - Since each part says "show that ...", if you get stuck on one step, you can always proceed and do the next step.

   a. The end result is that X ~ Geometric (p = 1/(θ+1)). Show that for this distribution,
      E(X) = θ and Var(X) = $\theta^2 + \theta$.

   b. Show that for this problem E(X) = E[E(X|Y)] and Var(X) = Var[E(X|Y)] + E[Var(X|Y)].

   c. Show that for Y ~ exponential(θ), the moment generating function is given by $M_Y(t) = (1-\theta t)^{-1}$ for $t < 1/\theta$.

   d. Give an argument to show that if Y ~ exponential(θ), then $E(Y^k) = k!\theta^k$ for k=0,1,2,...

   e. By conditioning on Y, show that the probability distribution of X, $p_X(x)$, can be given by ${}^{E(e^{-Y}Y^x)}/_{x!}$.

   f. Show that for Y ~ exponential(θ) and x=0,1,2,... , $E(e^{-Y}Y^x) = [{}^{\theta}/_{(\theta+1)}]^{x+1}\,[{}^{x!}/_{\theta}]$.

   g. Put these together to show that X ~ geometric(p = 1/(θ +1)).

2. Let $Y_1,\ldots,Y_n$ be i.i.d. from the Uniform U(α,β) distribution with parameter space $-\infty < \alpha < \beta < \infty$. We define the following statistics: $\bar{Y}$ and S^2 denote the sample mean and (unbiased) variance, $Y_{(1)}$ and $Y_{(n)}$ denote the sample minimum and maximum, and R denotes the sample range, $R = Y_{(n)} - Y_{(1)}$.

   a. Find formulas for the method of moments estimators of α and β.

   b. For observed values $y_1,\ldots,y_n$, find the likelihood function. Be clear about the domain and where it is zero.

   c. Give the likelihood function if n=9 and the observed values happen to be (.1,.2,.3,.4,.5,.6,.7,.8,.9).

   d. Find sufficient statistics for α and β.

   e. Show that the maximum likelihood estimators for α and β are $Y_{(1)}$ and $Y_{(n)}$, respectively. Are they functions of the sufficient statistics?

   f. Consider the likelihood ratio test for H_0: (α=0 and β=1) versus the alternative that one or both of these equations is false. Show that the likelihood ratio test statistic $\lambda = R^n$ (Range to the n^{th} power) if the data lies entirely in the interval [0,1] and it is zero otherwise.

   g. Use the data in part c and the large sample approximation (yes, even though n is only 9) to perform the likelihood ratio test. Use level of significance .10 and state your decision.

3. A bag has 3 coins in it: a fair coin, a coin biased in favor of heads such that heads is thrice as likely as tails, and a two-headed coin. One of these coins is drawn at random and tossed n times. Let F, B, and T denote the events that the chosen coin is fair, biased, and two-headed respectively. Assume that each of the 3 coins is equally likely to be drawn from the bag and that outcomes in different tosses of any coin are independent of each other.

(a) Let A_n be the event that all n tosses turn up heads. Calculate $P(F|A_n)$, $P(B|A_n)$, and $P(T|A_n)$.

(b) Find $\lim_{n\to\infty} P(B|A_n)$ and

$$\lim_{n\to\infty} \frac{P(B|A_n)}{P(F|A_n)}.$$

(c) Find

$$\lim_{n\to\infty} \frac{P(F|A_n)}{P(T|A_n)}.$$

What is the smallest value of n such that $P(F|A_n) < 0.01P(T|A_n)$.

(d) Let C_n be the event that all n tosses turn up tails. Calculate $P(F|C_n)$, $P(B|C_n)$, and $P(T|C_n)$.

(e) Find

$$\lim_{n\to\infty} \frac{P(B|C_n)}{P(F|C_n)}.$$

What is the smallest value of n such that $P(B|C_n) < 0.01P(F|C_n)$?

4. Let $X_1, \ldots, X_n$ be a random sample from a uniform $U(0, \theta)$ distribution, where $\theta > 0$.

(a) Are $X_{(n)}$ and $\left(\frac{X_{(n)}}{X_{(n-1)}}, \frac{X_{(n-1)}}{X_{(n-2)}}, \cdots, \frac{X_{(2)}}{X_{(1)}}\right)$ independent? Explain your reasoning.

(b) Let $g(\theta) = \mathrm{Var}_\theta(X_1)$. Find the maximum likelihood estimator of $g(\theta)$. Also determine the bias, the variance, and the mean squared error of the maximum likelihood estimator of $g(\theta)$.

(c) Let $g(\theta) = \mathrm{Var}_\theta(X_1)$. Find the UMVU estimator of $g(\theta)$. Also determine the bias, the variance, and the mean squared error of the UMVU estimator of $g(\theta)$.

(d) Compare the performance of the UMVU estimator in part (c) and the maximum likelihood estimator in part (b) in terms of mean squared error. Which estimator is better? Explain your reasoning.

(e) Derive the MP test at level α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.

(f) Calculate the power of the MP test in part (e).

(g) Let $\theta = 1$. Find the distribution of $X_{(l)} - X_{(k)}$, where $1 \le k < l \le n$. In addition, find $E(X_{(l)} - X_{(k)})$ and $\mathrm{Var}(X_{(l)} - X_{(k)})$.