M.S. and M.A. Comprehensive Analysis Exam

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To get full credit you must show all your work.
This exam contains 6 real analysis and 6 complex variables questions.

Real Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. (a) Define the supremum of a bounded set $A \subset \mathbb{R}$.

   (b) Let $A = \left\{ \frac{1}{n} - \frac{1}{m} : n, m = 1, 2, 3, \dots \right\}$. Find $\sup A$. Prove your claim.

2. (a) Define a uniformly continuous function on $\mathbb{R}$.

   (b) Use the definition to show that $f(x) = \dfrac{1}{x+1}$ is uniformly continuous on $[0, \infty)$.

3. (a) Let f be a bounded function on $[a, b]$. Define the Riemann integral $\int_a^b f$.

   (b) Use the definition of Riemann integration to compute $\displaystyle\int_{-1}^{1} f$ for $f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ x+1 & 0 < x \le 1 \end{cases}$.

4. Let $x_n \ge 0$ for all n and suppose that $\lim\limits_{n \to \infty} (-1)^n x_n$ exists. Show that $\lim\limits_{n \to \infty} x_n = 0$.

5. Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$ be a continuous function. Show that if X is connected, then $f(X)$ is connected.

6. Suppose that (X, d) is a complete metric space and $\{E_j : j \in \mathbb{N}\}$ is a collection of nonempty compact sets in X such that $E_1 \supset E_2 \supset E_3 \supset \cdots$. Show that $\displaystyle\bigcap_{j=1}^{\infty} E_j \neq \emptyset$.

Complex Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. Find the image of the square $\{x+iy : 0 \leq x \leq \ln 2, 0 \leq y \leq \pi\}$ under the mapping e^{2z}.

2. Let C denote the unit circle with counter-clockwise orientation. Compute the integral
$$\oint_C \mathrm{Re}(z)\bar{z}dz.$$

3. Expand $f(z) = \dfrac{1}{z^2-1}$ in a Laurent series valid for $\{z \in \mathbb{C} : 0 < |z-1| < 2\}$.

4. Evaluate the integral $\displaystyle\int_0^\infty \frac{1}{(1+x^2)^2}dx$.

5. Find an entire function whose imaginary part is $v(x,y) = x^2 - y^2 + 2$.

6. (a) State Maximum Modulus Principle for analytic functions.

   (b) Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $u : \mathbb{D} \to \mathbb{R}$ be a harmonic function. Show that if u attains a maximum (or minimum) in $\mathbb{D}$ then it is constant. (Hint: use Maximum Modulus Principle for analytic functions.)