MS Comprehensive EXAM
DIFFERENTIAL EQUATIONS
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This exam has two parts, ordinary differential equations and partial differential equations. Choose four problems in each part.

Part I: Ordinary Differential Equations

1. Find the solution of the initial value problem on $(0, \infty)$.

$$y''(x) - \frac{3}{x}y'(x) + \frac{29}{x^2}y(x) = 0, \quad y(1) = 3, \quad y'(1) = -2.$$

2. Find three linearly independent solutions of the equation and show that these solutions are linearly independent.

$$y''' - 3y'' + 3y' - y = 0.$$

3. Draw the phase portrait of the differential equation on the half plane $x \geq 0$. Find the smallest v_0 such that the solution $x(t)$ of the equation with the initial conditions $x(0) = 0, \dot{x}(0) = v_0$ satisfies $\lim_{t\to\infty} x(t) = \infty$.

$$\ddot{x} = -\frac{12}{(3+x)^2}$$

4. Compute eigenvalues and eigenfunctions of the following Sturm-Liouville problem.

$$y''(x) + \lambda y(x) = 0; \quad y(0) = y(2\pi), y'(0) = y'(2\pi).$$

5. Solve the nonhomogeneous linear system for $x \in R^2$ with the initial condition.

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ e^{2t} \end{bmatrix}, \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

6. Let $f(t)$ and $g(t)$ be two linearly independent solutions of the homogeneous equation $y''(t) + p(t)y'(t) + q(t)y(t) = 0$ on an open interval, where $p(t), q(t)$ are continuous functions. Show that 1) the roots of $f(t), g(t)$ are isolated; 2) between any two consecutive roots of $f(t)$ there is exactly one root of $g(t)$.

7. Consider a homogeneous n-th order linear ODE ($n \geq 1$):

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0,$$

on an interval I of the real line with real or complex valued continuous function p_{n-1}. Let $y_1, \ldots, y_n$ be n real or complex valued solutions of the given n-th order ODE and consider the Wronskian $W(y_1, \ldots, y_n)(x)$, $x \in I$. Show that the Wronskian satisfies the following formula (Abel's formula):

$$W(y_1, \ldots, y_n)(x) = W(y_1, \ldots, y_n)(x_0) \exp\left(-\int_{x_0}^{x} p_{n-1}(\xi)\, d\xi\right), \quad x \in I$$

for every point x_0 in I.

8. Show that

$$\dot{x} = -x + 2y^3 - 2y^4, \quad \dot{y} = -x - y + xy,$$

has no periodic solutions, by choosing a, m, n so that $V(x, y) = x^m + ay^n$ is a Lyapunov function.

9. Consider the system

$$\dot{x} = x^2 - y - 1, \quad \dot{y} = y(x - 2).$$

Show that there are three fixed points and classify them. By considering the three straight lines through pairs of fixed points, show that there are no closed orbits and sketch the phase portrait.

Part II: Partial Differential Equations

1. Find a solution of the initial value problem for the heat equation. Use the formula $u(x,t) = \frac{1}{\sqrt{4\pi at}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4at}} f(y)dy$ for the heat equation and the formula $\int_{-\infty}^{\infty} e^{-cx^2} dx = \sqrt{\pi/c}$.

$$\begin{cases} u_t(x,t) = 9u_{xx}(x,t) & \text{on } -\infty < x < \infty,\ t > 0; \\ u(x,0) = e^{-4x^2} & \text{for } -\infty < x < \infty. \end{cases}$$

2. Find all radially symmetric solutions of $\Delta u + 9u = 0$ in R^3.

3. Solve the initial-boundary value problem for the wave equation.

$$\begin{cases} u_{tt}(x,t) = 9u_{xx}(x,t) & \text{on } 0 \le x \le \pi, t > 0; \\ u(0,t) = 0, \quad u(\pi,t) = 0 & \text{for } t > 0; \\ u(x,0) = 3\sin(x) - \frac{3}{4}\sin(2x) - 2\sin(3x), & \\ u_t(x,0) = 0 & \text{for } 0 \le x \le \pi. \end{cases}$$

4. Let $u(x,t)$ be a function with continuous second derivatives that satisfies the following initial-boundary conditions. Show that $u(x,t)$ is identically zero on the half stripe domain defined in the problem.

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & \text{on } 0 < x < \pi,\ t > 0; \\ u(x,0) = u_t(x,0) = 0 & \text{for } 0 < x < \pi; \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0. \end{cases}$$

5. For the $u(x,y)$ as below, find $v(x,y)$ satisfying the Cauchy-Riemann equations $\partial u/\partial x = \partial v/\partial y, \partial u/\partial y = -\partial v/\partial x$ and the condition $v(0,0) = 0$.

$$u(x,y) = x + y + 3x^2y - y^3$$

6. Solve the initial value problem.

$$\begin{cases} u_x + u_y = u \\ u(x, 0) = x^2 + 1. \end{cases}$$

7. Consider the following evolutionary PDE on the real line $(-\infty, +\infty)$ for an unknown function $u(x,t)$, which is assumed to be sufficiently differentiable:

$$u_t = uu_x + u_{xxx}.$$

Show that if one is looking for traveling wave solutions $u(x,t) = f(x - ct)$, the given PDE can be reduced to a nonlinear ODE (but do not try to solve the corresponding ODE).

8. Consider the equation $yu_x - xu_y = 0$. Find curves an initial conditions along these curves for which this problem has a unique solution, no solution or infinitely many solutions.

9. Classify the equation

$$xu_{xx} - 4u_{xt}$$

in the domain $x > 0$. Find the general solution of this equation by the nonlinear change of variables $\tau = t$ and $\xi = t + 4\ln(x)$.

10. Consider the heat flow in a thin circular ring of unit radius that is insulated along its lateral surface. The temperature distribution in the ring can be described by the standard one-dimensional diffusion equation, where x represents the arc length along the ring. The shape of the ring means that you need to consider periodic boundary conditions:

$$u(-\pi, t) = u(\pi, t), \quad u_x(-\pi, t) = u_x(\pi, t).$$

Solve this problem for a general initial condition $u(x, 0) = \phi(x)$ which is assumed to be 2π periodic, i.e. $\phi(x + 2\pi k) = \phi(x)$ for all k integers.