Department of Mathematics
The University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

April 16, 2016

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
Books, notes, and calculators may be used.
This is a three hour test.

1. (20 points) A supplier of kerosene has a weekly demand Y possessing a probability density function given by

$$f(y) = \begin{cases} y, & 0 \le y \le 1, \\ 1, & 1 < y \le 1.5, \\ 0, & \text{elsewhere}, \end{cases}$$

with measurements in hundreds of gallons. The supplier's profit is given by $U = 10Y - 4$.

(a) (10 points) Find the probability density function for U.

(b) (10 points) Use the answer to part (a) to find $E(U)$.

2. (30 points) Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the Poisson distribution with mean λ.

(a) (10 points) Find the MLE $\hat{\lambda}$ of λ.

(b) (10 points) Find the expected value and variance of $\hat{\lambda}$.

(c) (5 points) Show that the estimator of part (a) is consistent for λ.

(d) (5 points) What is the MLE for $P(Y = 0) = e^{-\lambda}$?

3. The number of customers Z that arrive at the bank on a certain day is distributed as a Poisson random variable, with (mean) parameter $\lambda > 0$. Each customer, independently of the others, either goes to the teller with probability p, or to the ATM, with probability $1 - p$ (with $0 < p < 1$). Let X be the number of customers that are going to the teller, and Y the number of customers going to the ATM.

(a) Find the probability distribution of X.

(b) Find the probability distribution of Y.

(c) Are X and Y independent? Explain your reasoning.

4. Let

$$
\begin{aligned}
Y_1 &= \beta_1 + \epsilon_2 \\
Y_2 &= 2\beta_1 - \beta_2 + \epsilon_2 \\
Y_3 &= \beta_1 + 2\beta_2 + +\epsilon_3,
\end{aligned}
$$

where ϵ_1, ϵ_2, and ϵ_3 are uncorrelated random variables with $E(\epsilon_i) = 0$ and $\mathrm{Var}(\epsilon_i) = \sigma^2$ for $i = 1, 2, 3$.

(a) Find the least squares estimates $(\hat{\beta}_1, \hat{\beta}_2)$ of (β_1, β_2).

(b) Find $\mathrm{Var}(\hat{\beta}_1)$ and $\mathrm{Var}(\hat{\beta}_2)$.

(c) Find $\mathrm{Cov}(\hat{\beta}_1, \hat{\beta}_2)$.

(d) Suppose further that $\sigma^2 = 1$ and $\epsilon_1, \epsilon_2, \epsilon_3$ are iid $N(0, 1)$ random variables. Find a complete and sufficient statistic for (β_1, β_2).