MA exam: Algebra, 22 April 2017

Directions: Do all of the six problems. Your best five answers will determine your grade. Give complete proofs — do not just quote a theorem.

Notation: $\mathbb{C}$ denotes the complex numbers, $\mathbb{R}$ the reals, and $\mathbb{Z}$ the integers.

Caveat: The examiners have made every effort to make sure that the problems are correct. However, if you are convinced that a problem contains a misprint, explain why you think so, formulate a correct and non-trivial version of the problem, and do that instead.

Part I: Group theory

1. This problem concerns S_{13}, the symmetric group on 13 letters, where we write our functions on the left. Let

$$g_1 = (1,3,9)(2,8,5,7,10,12,13,4,6,11)$$
$$g_2 = (2,5,14,9,12)(3,4,7,11,6,8).$$

(a) Compute the orders $|g_1|$ and $|g_2|$.

(b) Compute the group element $g_1 g_2 g_1^{-1}$.

(c) Are g_1 and g_2 conjugate in S_{13}? Prove your assertion.

2. This problem concerns A_4, the alternating group on four letters.

(a) Show that A_4 contains a normal subgroup of order 4.

(b) Prove of disprove: There is a homomorphism from A_4 onto $\mathbb{Z}/3\mathbb{Z}$, the group of integers modulo 3.

(c) Prove of disprove: There is a homomorphism from A_4 onto $\mathbb{Z}/4\mathbb{Z}$.

(d) Prove of disprove: There is a homomorphism from A_4 onto S_3, the symmetric group on 3 letters.

Part II: Ring theory

3. Prove that a finite integral domain is a field.

4. Consider $\mathbb{Z}[x]$, the ring of polynomials having integer coefficients. Either prove that $\mathbb{Z}[x]$ is a principal ideal domain or prove that it is not.

Part III: Linear algebra

5. Let V be a finite dimensional vector space over $\mathbb{C}$ and $T : V \to V$ be a linear transformation satisfying the equation $T^4 = I$.

(a) Prove that T can be represented by a diagonal matrix.

(b) Give an example to show that if V is a finite dimensional vector space over $\mathbb{R}$, and T is as above, then T need not be diagonalizable.

6. Consider the 4×4 real matrix

$$A = \begin{pmatrix} 2 & 1 & -1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(a) Find J, the Jordan canonical form of A.

(b) Find an invertible matrix P so that $AP = PJ$.