MS COMPREHENSIVE EXAM
Real and Complex Analysis
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This exam has two parts, (A) Real Analysis and (B) Complex Analysis. Do any four of the six problems in part A and any three of the five problems in part B. Clearly indicate which problems in each part are to be graded. Show the details of your work.

Part A: Real Analysis (Do any 4 of the 6 problems.)

1. (a) Suppose that a_n is a bounded, real sequence. Define $\limsup_n a_n$.

   (b) Suppose that a_n and b_n, $n \in \mathbb{N}$ are two bounded real sequences. Show that

   $$\limsup_n (a_n + b_n) \leq \limsup_n a_n + \limsup_n b_n$$

   (c) Further show, by example, that strict inequality $\limsup_n a_n + b_n < \limsup_n a_n + \limsup_n b_n$ is possible.

2. Suppose (f_n) is a sequence of functions converging uniformly to zero on given interval $[a, b]$. (We are not assuming the f_n continuous!) Let (x_n) be any convergent sequence of points in $[a, b]$. Show that $\lim_{n\to\infty} f_n(x_n) = 0$. Using an example show that this is false if $f_n \to 0$ only pointwise. Suppose instead now that (f_n) is a sequence of functions on an interval $[a, b]$, with the property that for any converging sequence of points (x_n) in $[a, b]$ we have $\lim_{n\to\infty} f_n(x_n) = 0$. Show that indeed the convergence of (f_n) to zero on $[a, b]$ is uniform.

3. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Evaluate the following limits with proof:

   $$\lim_{n\to+\infty} \int_0^1 x^n f(x)\, dx \qquad \lim_{n\to+\infty} n \int_0^1 x^n f(x)\, dx$$

4. Suppose that $f : (0, 1) \to \mathbb{R}$ is continuous. Prove or disprove.

   (a) If f is uniformly continuous then f is bounded.

   (b) If f is bounded then f is uniformly continuous.

   (c) $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

5. Consider the map $id : C_{\max} \to C_{int}$ sending any f to itself, where $C_{\max}$ is the metric space $C^0([a,b],\mathbb{R})$ of continuous real valued functions equipped with the maximum metric $d_{\max}(f,g) = \max|f(x) - g(x)|$ and C_{int} is again the same space but equipped with the metric

$$d_{int}(f,g) = \int_a^b |f(x) - g(x)|\, dx.$$

Show that id is a continuous linear bijection but its inverse is not continuous.

6. a) State what it means for a function $f : \mathbb{R}^2 \to \mathbb{R}$ to be differentiable at a point. Consider now the function

$$f(x,y) = \begin{cases} \frac{x^2y^5}{(x^2+y^2)^3} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

b) Is f continuous at $(0,0)$? c) Does it admit directional derivatives along any direction at $(0,0)$? d) Is it differentiable at $(0,0)$?

Part B: Complex Analysis (Do any 3 of the 5 problems)

1. a) State a version of Rouché theorem. b) Let $a \in \mathbb{C}$, $|a| > e$. Use Rouché theorem to prove that the equation $e^z = az^n$ has n solutions (not necessarily distinct) in the open unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$.

2. Let $H = \{z = x + iy : y > 0, x \in \mathbb{R}\}$ denote the upper halfplane. Determine the image of H under the map $z \mapsto \frac{1}{z+1+i}$ and sketch the image.

3. Let Γ denote the positively oriented unit circle. Evaluate
$$\int_\Gamma \frac{1}{z^5 + 3z^2 + 5} dz.$$

4. Find Laurent expansions for
$$f(z) = \frac{4z}{(z+1)(z-3)}$$
valid in (a) $\{z : 1 < |z| < 3\}$; (b) $\{z : |z| > 3\}$.

5. Use the residue theorem to evaluate the integral. (Do Part a or Part b and not both.)

   (a) $\int_0^\infty \frac{1}{(x^2+1)^2}\, dx$

   (b) $\int_0^\pi \frac{1}{2+\cos\theta}\, d\theta$