MS/MA COMPREHENSIVE EXAM
DIFFERENTIAL EQUATIONS
Spring 2017

Alessandro Arsie and Ivie Stein

This exam has two parts, (A) ordinary differential equations and (B) partial differential equations. Do any three of the five problems in part A and any three of the five problems in part B. Clearly indicate which three problems in each part are to be graded. Show the details of your work.

Part A: Ordinary Differential Equations

(1) Consider the initial value problem

$$u'' - (1 - u^2)u' + u = 0,\ u(0) = -1,\ u'(0) = 1.$$

(a) Convert the initial value problem to a first order system of ordinary differential equations with initial conditions.

(b) Apply one step of Euler's numerical method for vectors with stepsize $h = .01$ to the first order system with initial conditions found in part (a) above.

(c) Use the results of part (b) above to estimate the values of $u(.01)$ and $u'(.01)$ in the original initial value problem.

(2) Let $A = \begin{pmatrix} 1 & 1 \\ 9 & 1 \end{pmatrix}$.

(a) Find all eigenvalues of A.

(b) For each eigenvalue of A, find the corresponding eigenvectors.

(c) Draw the phase portrait of $\dot{x} = Ax$ nearby the origin in $\mathbb{R}^2$.

(d) Find the general solution to $\dot{x} = Ax$.

(3) Let $F(s) = \mathfrak{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}dt$. Suppose $\lim_{t\to 0}\left(\frac{f(t)}{t}\right)$ exists and is finite. Prove

$$\mathfrak{L}\left\{\frac{f(t)}{t}\right\}(s) = \int_{u=s}^{u=\infty} F(u)du,$$

where $F(u) = \mathfrak{L}\{f(t)\}(u) = \int_0^\infty f(t)e^{-ut}dt.$

(4) Consider the second order equation

$$y'' + p(x)y' + q(x)y = 0$$

on an open interval I where p and q are continuous.

(a) State the Bolzano-Weierstrass Theorem.

(b) Let y be a non-zero solution. Prove that all zeros of y are isolated.

(c) Let y_1 and y_2 be two linearly independent solutions. Prove that between any two consecutive zeros of y_1 there is exactly one zero of y_2.

(5) Compute eigenvalues and eigenfunctions of the following Sturm-Liouville problem (the eigenvalues are expressed in terms of the roots of a transcendental equation that can not be solved exactly, so estimate graphically the position of these roots):

$$y''(x) + \lambda y(x) = 0, \ 0 < x < \pi,$$
$$y'(0) - y(0) = 0, \ y(\pi) = 0.$$

Work completely any three of the five problems.

Part B: Partial Differential Equations

(1) Suppose $f^{(4)}$ is continuous on $[x_0 - h, x_0 + h]$, $h > 0$.

(a) Show that

$$f''(x_0) = \frac{1}{h^2}\left[f(x_0 - h) - 2f(x_0) + f(x_0 + h)\right] + E$$

where

$$E = \frac{-h^2}{12} f^{(4)}(\xi),$$

for some ξ, where $x_0 - h < \xi < x_0 + h$.

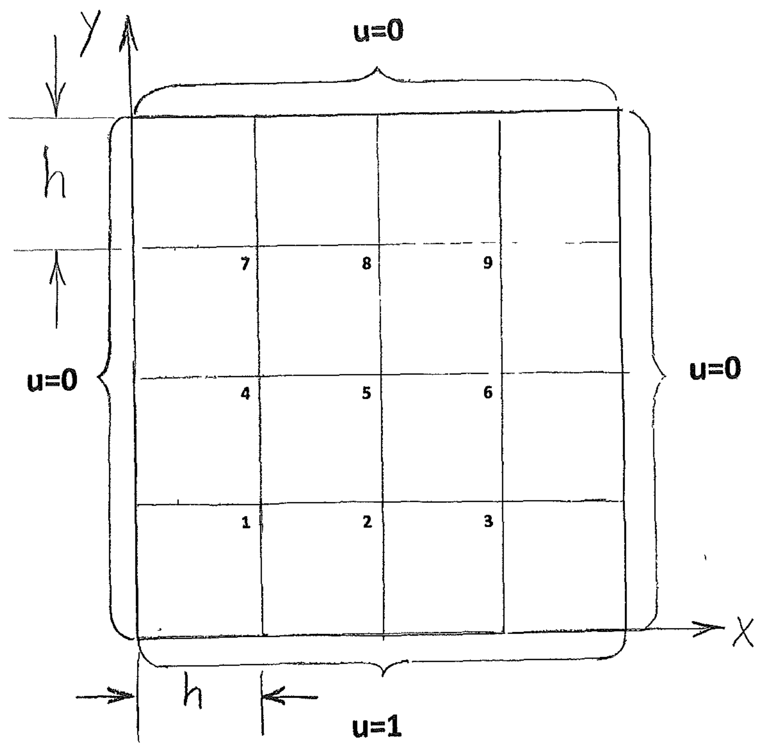
(b) Use part (a) above to obtain for small h, $h > 0$,

$$\frac{\partial^2 u}{\partial x^2}(x_0, y_0) \cong \frac{u(x_0 - h, y_0) - 2u(x_0, y_0) + u(x_0 + h, y_0)}{h^2}.$$

$$\frac{\partial^2 u}{\partial y^2}(x_0, y_0) \cong \frac{u(x_0, y_0 - h) - 2u(x_0, y_0) + u(x_0, y_0 + h)}{h^2}.$$

Assume that the fourth partial derivatives of u are continuous.

(c) Apply the approximations in part (b) to obtain the following linear system of equations that approximates the solution to the Dirichlet problem given below:
Dirichlet problem:



linear system:

$$\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where u_i approximates the value of the solution to the Dirichlet problem at point i in the diagram, $i = 1, 2, \cdots, 9$. Assume h is small, $h > 0$. Do not attempt to solve the linear system.

(2) Solve the following Initial/Boundary value problem for the heat equation in one dimension using separation of variables (since the initial value u_0 is unspecified, just write the formula for the Fourier coefficients). Here u_0 is a given continuous function on $[0, \pi]$.

$$u_t - u_{xx} = 0 \text{ in } (0, \pi) \times (0, +\infty)$$
$$u(x, 0) = u_0(x), \; x \in (0, \pi)$$
$$u(0, t) = u(\pi, t) = 0, \; \forall t \in (0, +\infty).$$

(3) Find the general solution $u(x, y)$ of

$$u_{xxy} = 1.$$

(4) (a) Prove the following identity:

$$\frac{1}{2} + \cos(u) + \cos(2u) + \cdots + \cos(nu) = \frac{\sin((n+\frac{1}{2})u)}{2\sin(\frac{u}{2})}$$

(b) Use part (a) to show

$$\int_{-\pi}^{\pi} \frac{\sin((n+\frac{1}{2})u)}{2\sin(\frac{u}{2})}\, du = \pi$$

(c) Show $\frac{\sin((n+\frac{1}{2})u)}{2\sin(\frac{u}{2})}$ is periodic with period 2π. We define function $f(x)$ to be *periodic* with *period* p if $f(x+p) = f(x)$ for all values of x.

(d) Let $f(x)$ be a piecewise smooth function with period 2π. Let

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx,\ k = 0, 1, 2, \cdots,$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx,\ k = 1, 2, \cdots,$$

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos(kx) + b_k\sin(kx)).$$

Show

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\left[\frac{1}{2} + \sum_{k=1}^{n}(\cos(kx)\cos(k\xi) + \sin(kx)\sin(k\xi)\right] d\xi,$$

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\left[\frac{1}{2} + \sum_{k=1}^{n}\cos(k(x-\xi))\right] d\xi, \text{ and}$$

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\,\frac{\sin((n+\frac{1}{2})(x-\xi))}{2\sin(\frac{(x-\xi)}{2})}\, d\xi.$$

(5) Find the Green function for the Laplace operator in the first quadrant in $\mathbb{R}^2$.