Department of Mathematics
The University of Toledo

Master of Science Degree
Comprehensive Examination
Probability and Statistical Theory

April 15, 2017

Instructions

Do all four problems.

Show all of your computations.
Prove all of your assertions or quote appropriate theorems.
Books, notes, and calculators may be used.
This is a three hour test.

1. (50 pts) Suppose that $\{Y_1, \cdots, Y_n\}$ are independent random variables and $Y_i \sim N(\beta x_i, 1)$, where $x_1, \cdots, x_n$ are fixed **known** constants, and β is an **unknown** parameter.

   a. (5 pts) Show that $\tilde{\beta} = \left(\sum_{i=1}^{n} Y_i\right) / \left(\sum_{i=1}^{n} x_i\right)$ is an unbiased estimator of β.

   b. (5 pts) Calculate $\mathsf{Var}\left(\tilde{\beta}\right)$.

   c. (5 pts) Find a sufficient and complete statistic for β.

   d. (5 pts) Find the maximum likelihood estimator (MLE) $\hat{\beta}$.

   e. (5 pts) Is the MLE $\hat{\beta}$ the uniformly minimum variance unbiased estimator (UMVUE)? Briefly explain.

   f. (5 pts) Calculate the Cramér-Rao Lower Bound. Does the MLE $\hat{\beta}$ reach it? Briefly explain.

   g. (5 pts) Show that $\sqrt{\sum_{i=1}^{n} x_i^2}\left(\hat{\beta} - \beta\right) \sim N(0, 1)$.

      Hint: $\hat{\beta}$ is a linear combination of independent random variables $\{Y_1, \cdots, Y_n\}$ with $Y_i \sim N(\beta x_i, 1)$.

   h. (5 pts) Find a uniformly most powerful (UMP) level α test for

   $$H_0 : \beta = 0 \quad \text{vs} \quad H_1 : \beta > 0.$$

      Write the rejection region R of this test using the quantile of a well-known distribution.

   i. (5 pts) Find the likelihood ratio test (LRT) statistic Λ for

   $$H_0 : \beta = 0 \quad \text{vs} \quad H_1 : \beta \neq 0.$$

   j. (5 pts) Describe how you use the LRT statistic Λ to test H_0 vs H_1; that is, how you calculate the p-value or the rejection region.

2. The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} 0.2, & \text{if } -1 < x \leq 0, \\ 0.2 + cx, & \text{if } 0 < x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find c.

(b) Find the distribution function of X.

(c) Find $P(X \geq 0.5 | X \geq 0.1)$.

(d) If $M_X(t)$ is the moment-generating function of X, find $M_X(0)$, $M'_X(0)$ and $M''_X(0)$.

3. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with density function

$$f(x;\theta) = \theta \exp(-\theta x), \qquad x > 0, \ \ \theta > 0.$$

(a) Find the maximum likelihood estimate of θ^2.

(b) Find the Fisher information about θ contained in $(X_1, \ldots, X_n)$. Also find the Cramér-Rao lower bound for the variance of an unbiased estimate of θ^2.

? (c) Find the UMVU estimator of θ^2. Does the UMVU estimator achieve the Cramér-Rao lower bound? Explain your reasoning.

(d) Suppose that we wish to test the null hypothesis $H_0 : \theta \leq 0.5$ versus the alternative hypothesis $H_1 : \theta > 0.5$. Find the most powerful critical region of size α.