# MA exam: Algebra — 21 April 2018

Do two problems from each of the three sections. Give complete proofs — do not just quote a theorem. Please indicate clearly which six problems you want graded.

## Part I: Group theory

**1.** If $G$ is a group and $H$ is a subgroup of $G$ of index $n$, show that $G$ contains a normal subgroup $K$ whose index in $G$ divides $n!$.

**2.** (a) If $G$ is a group which contains only a finite number or subgroups, show that $G$ is finite.

(b) Describe all groups $G$ which containing no proper subgroups.

(c) Describe all groups $G$ which contain exactly one proper non-trivial subgroup.

**3.** Let $G$ be a finite $p$-group and let $H$ be a normal subgroup of $G$ of order $p$. Show that $H$ is contained in the center of $G$.

## Part II: Ring theory

**4.** Let $\mathbb{Z}_3$ be the field with 3 elements. Find all monic irreducible polynomials of degree 3 in the ring $\mathbb{Z}_3[x]$.

**5.** Let $\mathbb{Z}$ be the ring of integers, $p$ a prime in $\mathbb{Z}$, and $\mathbb{Z}_p$, the field of $p$ elements. Let $x$ be an indeterminate, and set

$$R_1 = \mathbb{Z}_p[x]/(x^2 - 2), \qquad R_2 = \mathbb{Z}_p[x]/(x^2 - 3).$$

Determine whether the rings $R_!$ and $R_2$ are isomorphic in each of the following cases: $p = 2, 5, 11$.

**6.** Let $\mathbb{F}$ be field, and let $R$ be the subset of $2 \times 2$ matrices over $\mathbb{F}$ which commute with the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(a) Prove the $R$ is a commutative ring.

(b) Prove that $R \cong \mathbb{F}[x]/I$, where $I$ is the ideal of $\mathbb{F}[x]$ generated by $x^2$.

## Part III: Linear algebra

**7.** Consider the $4 \times 4$ real matrix

$$A = \begin{pmatrix} 2 & 1 & -1 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(a) Find $J$, the Jordan canonical form of $A$.

(b) Find an invertible matrix $P$ so that $AP = PJ$.

**8.** (a) Let $V$ and $W$ be vector spaces over a field $\mathbb{F}$, and let $T$ be a lineard operator from $V$ into $W$. Suppose that $V$ is finite-dimensional. Prove $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V)$.

(b) Let $S$ be the linear operator defined on the space of $3 \times 3$ real matrices given by

$$S(A) = A - A^t,$$

where $A^t$ denotes the transpose of the matrix $A$. Determine $\operatorname{rank}(S)$.

**9.** Let $A$ be a $3 \times 3$ matrix over the field $\mathbb{R}$ of real numbers and suppose that $\operatorname{tr}(A) = 6$, $\operatorname{tr}(A^2) = 14$ and $\det(A) = 6$. Here $\operatorname{tr}(A)$ and $\det(A)$ denote the trace and determinant of $A$. Prove that $A$ is similar to the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$