M.S. and M.A. Comprehensive Analysis Exam

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To get full credit you must show all your work.
This exam contains 6 real analysis and 6 complex variables questions.

Real Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. (a) Define the supremum of a bounded set $A \subset \mathbb{R}$.

   (b) Let A and B be two bounded set in $\mathbb{R}$. Show that $\sup(A+B) = \sup A + \sup B$ where $A + B = \{a + b : a \in A, b \in B\}$.

2. (a) Let f be a bounded function on $[a,b]$. Define the Riemann integral $\int_a^b f$.

   (b) Use the definition of Riemann integration to compute $\displaystyle\int_0^1 (2x-1)$.

3. Decide whether the following series converge and justify your answers:

$$\sum_{n=2}^{\infty} \frac{n-2}{n^2-2}, \quad \sum_{n=1}^{\infty} \frac{n!}{n^n}, \quad \sum_{n=1}^{\infty} \frac{\ln n}{n}.$$

4. (a) State Rolle's Theorem.

   (b) Examine whether the hypotheses and conclusions of Rolle's Theorem hold for:

      i. $f(x) = 1 - |x|$ defined on the interval $[-1, 1]$,

      ii. $f(x)$ defined by $x \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$ on the interval $[\frac{-1}{2\pi}, \frac{1}{2\pi}]$.

   (c) If $a_0, a_1, ...a_n$ are real numbers such that $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0$. Prove that the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0 \tag{1}$$

   has a solution between 0 and 1.

5. Define uniform continuity for real-valued functions on metric spaces. Let (X, d) be a compact metric space and $f : X \to \mathbb{R}$ be a continuous function. Show that f^2 (pointwise product) is uniformly continuous.

6. (a) Let $X = (0, \infty) \subset \mathbb{R}$ and $\rho(x, y) = |\ln(x/y)|$. Use the definition of metric to prove that ρ is a metric on X.

   (b) Let $(X, d_X), (Y, d_Y)$ be two metric spaces and $f : X \to Y$ be a continuous function. Show that if $A \subset X$ is connected then $f(A)$ is connected.

Complex Analysis

100% will be obtained for complete answers to four questions. Indicate clearly which four questions you wish to be graded.

1. The symbol $(1 + i)^{(1+i)}$ is multi-valued. Find all the values.

2. Determine from the definition or by means of Cauchy-Riemann equations the points at which the function $f(z) = z\bar{z}$ is analytic (holomorphic).

3. (a) Let C be a piecewise smooth closed contour in $\mathbb{C}$. Suppose that $f(z)$ is an entire function and $a, b \in \mathbb{C} \setminus C$. Find all possible values for the integral

$$\oint_C \frac{f(z)}{(z-a)(z-b)} dz.$$

   (b) Evaluate the integral $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + \cos\theta}$ for any real number $a > 1$.

4. Let

$$f(z) = \frac{z^3 + 2z^2 + 4}{(z-1)^3}.$$

   Find the Laurent series expansion for $f(z)$ about the singular point $z = 1$. Find the region for which this expansion is valid.

5. Find an entire function whose real part is $e^x(x\cos y - y\sin y)$.

6. Let U be the upper half plane. Find all possible analytic functions $f : \mathbb{C} \to U$. Prove that you have found all such functions.