Probability and Statistical Theory

MS Comprehensive Examination

April 14, 2018

Instructions:

Show all of your computations.
Prove all of your assertions or quote the appropriate theorems.
Books, notes, and calculators *may be used*.

You have three hours.

1. [25 pts] Suppose that A_1 and A_2 are independent uniform random variables on $[0,1]$. Let $X = \max\{A_1, A_2\}$, $Y = \min\{A_1, A_2\}$ and $Z = A_1 + A_2$. Compute the following:

(a) The probability density function f_X

(b) The expectation $E(X)$.

(c) The conditional expectation $E(Y|X)$ (Hint: $P(Y \leq y, X \leq x) = P(X \leq x) - P(Y > y, X \leq x)$)

(d) The covariance $\text{Cov}(X, Y)$

(e) The probability density function f_Z.

(f) The expectation $E(Z)$ and variance $Var(Z)$.

(g) The covariance $\text{Cov}(A_1, Z)$ and correlation $\rho(A_1, Z)$

2. [25 pts] Let us have two independent random samples: $X_1, ..., X_n$ is a sample from $N(\mu_x, \sigma_x^2)$, and $Y_1, ..., Y_m$ is a sample from $N(\mu_y, \sigma_y^2)$

(a) Write down a joint pdf for $\{X_1, ..., X_n, Y_1, ..., Y_m\}$

(b) Find a 4-dimensional sufficient statistic.

(c) Find the MLE of σ_x^2 and σ_y^2

(d) Assume $\sigma_x^2 = \sigma_y^2 = \sigma^2$. Find the MLE for σ^2.

(e) Find a LR test statistic for testing $H_0 : \sigma_x^2 = \sigma_y^2$

3. (50 points, 5 points each) $X_1, \cdots, X_n \overset{iid}{\sim} f(x|\theta) = \frac{3x^2}{\theta} \exp\left(-\frac{x^3}{\theta}\right)$ for $x \geq 0$ and $\theta > 0$.

a. Find the distribution of $Y = \sum_{i=1}^{n} X_i^3$.

b. Find $\mathsf{E}Y$ and $\mathsf{Var}(Y)$.

c. Find a complete and sufficient statistic for θ.

d. Find the maximum likelihood estimator $\hat{\theta}_{\text{MLE}}$ for θ.

e. Explain whether $\hat{\theta}_{\text{MLE}}$ is the UMVUE. If it is not, find the UMVUE $\tilde{\theta}$.

f. Calculate the variance of the UMVUE $\tilde{\theta}$.

g. Calculate the Cramér-Rao Lower Bound. Does the UMVUE $\tilde{\theta}$ reach it?

h. Find a UMP level α test $\phi(T)$ for

$$H_0 : \theta = \theta_0 \text{ vs } H_1 : \theta > \theta_0.$$

Write the rejection region R of this test using the test statistic T and the quantile of a well-known distribution.

Consider $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$.

i. Explain why a UMP level α test does not exist.

j. Find the likelihood ratio test statistic Λ. Using the quantile of a well-known distribution, write the rejection region R of the likelihood ratio test so that its level is α.